







MODERN MATHEMATICS  
FOR  
T. C. MITS



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FOR  
T. C. MITS

*The Celebrated  
Man in the Street*

Drawings by  
HUGH GRAY LIEBER

Words by  
LILIAN R. LIEBER

London

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To  
My Wife )



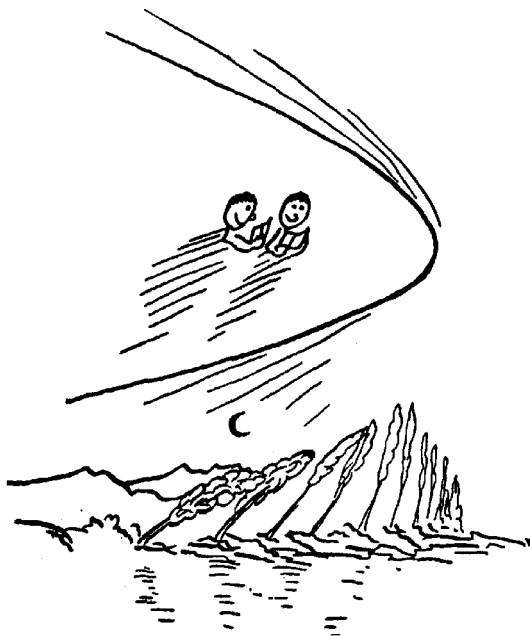
THIS BOOK IS PRODUCED IN  
COMPLETE CONFORMITY WITH THE  
AUTHORIZED ECONOMY STANDARDS

Printed in Great Britain by

## PREFACE

This is not intended to be  
free verse.

Writing each phrase on a separate line  
facilitates rapid reading,  
and everyone  
is in a hurry  
nowadays.







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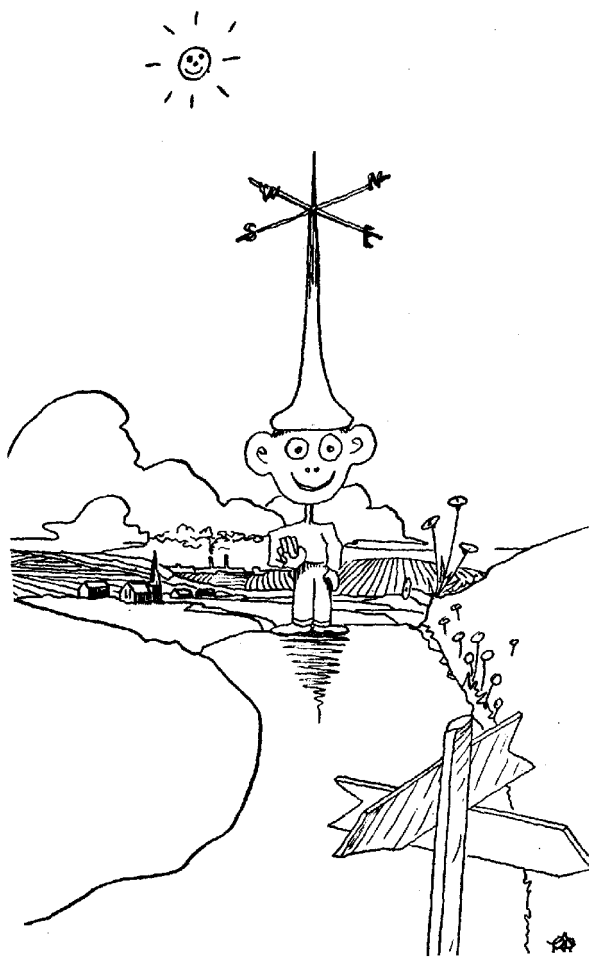


## INTRODUCING THE HERO—T. C. MITS

This introduces the Hero:

T.	C.	M	I	T	S
h	e	a	n	h	t
e	l	n		e	r
	e				e
	b				e
	• r				t
	a				
	t				
	e				
	d				

T. C. is born and gets  
an education of some kind—  
perhaps college,  
perhaps “the school of hard knocks.”  
In any case  
he tries to figure out  
how best to “get along.”  
He picks up a lot of  
contradictory information:



"The past is antiquated,  
you must be progressive."

"The past is wonderful,  
the new-fangled fads are  
a sign of decadence."

"Science will save us from  
Superstition and Fraud."

"Science is the greatest menace  
yet invented by man."

"Fifty million people can't be wrong."

"Some races are always wrong."

"Be practical, learn a vocation,  
don't waste your time on  
Mathematics and Art."

"Why be a narrow, practical farmer  
all your life,  
get out and learn some theory,  
and find out how  
to do things in a better way."

And so on and so on.

He is naturally confused by all this,  
and very much hemmed in.  
He becomes not only  
Mits in name,

but has mits on his fingers  
and mits on his toes,  
and is generally "mitsified"  
in the brain.

This book is an attempt  
to get a bird's-eye view  
of T. C.'s predicament,  
and to look for  
a possible egress.

To do this  
VIVIDLY,  
we use pictures whenever possible.  
And  
to do it  
CLEARLY,  
we use the clearest language  
man has invented:  
Mathematics.

Oh, we know you do not like  
Mathematics,  
but  
we promise not to  
use it as an instrument of torture,  
but to show  
what bearing it may have  
on the contradictory advice  
mentioned above,  
as well as on such things as:

Democracy  
Freedom and License  
Pride and Prejudice  
Success  
Isolationism  
Preparedness  
Tradition  
Progress  
Idealism  
Common Sense  
Human Nature  
War  
Self-reliance  
Humility  
Tolerance  
Provincialism  
Anarchy  
Loyalty  
Abstract Art  
and so on.

Now and then we shall point out  
a "Moral."  
But please do not think  
we are being didactic  
and preaching to the reader:

the fact is that we are really  
talking to ourselves,  
for we, along with millions of others,  
are T. C. himself.





**PART I**  
**THE OLD**



## I. FIFTY MILLION PEOPLE CAN BE WRONG

Let us begin with  
a very simple question:  
suppose you had the choice of  
the following two jobs:

Job 1: Starting with an  
annual salary of \$1000,  
• and a \$200 increase every year.

Job 2: Starting with a  
semiannual salary of \$500,  
and an increase of  
\$50 every 6 months.

In all other respects,  
the two jobs are exactly alike.

Which is the better offer  
(after the first year)?

Think carefully and  
decide on your answer

**BEFORE TURNING THIS PAGE.**

Did you say Job 1 is better?  
 And did you reason as follows?  
 Since Job 2 has an increase  
 of \$50 every 6 months,  
 it must have an annual increase of \$100  
 and therefore it is not as good  
 as Job 1 which has  
 an annual increase of \$200.

Well, you are wrong!  
 For, examine carefully  
 the earnings written out below:

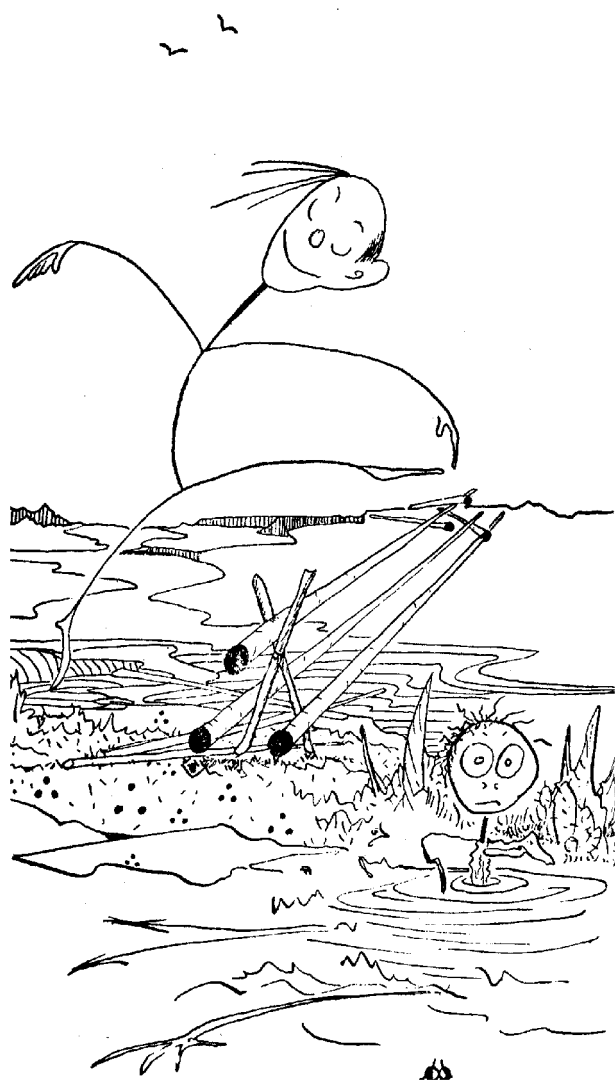
		1st half of year	2nd half of year	total for the year
1st year	{ Job 1	\$500	\$500	\$1000
	{ Job 2	500	550	1050
2nd year	{ Job 1	600	600	1200
	{ Job 2	600	650	1250
3rd year	{ Job 1	700	700	1400
	{ Job 2	700	750	1450
4th year	{ Job 1	800	800	1600
	{ Job 2	800	850	1650
etc., etc., etc.				

Note that:

- (1) Job 1 pays \$200 more each year  
than it did the previous year.
- (2) Job 2 pays \$50 more every  
half-year than it did during  
the previous half-year.

All this is in accordance with  
the promises originally made,  
and yet  
Job 2 brings in \$50 more every year  
than Job 1 does.  
And you can easily see that  
this will continue to be true  
no matter what number of years  
is considered.

You are probably surprised.  
But don't be discouraged,  
for you are in plenty of  
good company.  
Try it on your friends,  
and you will find that,  
unless they have heard it before,  
they will probably make  
the same mistake that you made.  
Fifty million people CAN be wrong!  
And this is entirely normal.  
But please do not come to  
the conclusion that  
Democracy is no good!  
For fifty million people  
do not HAVE to be wrong!  
They may be wrong when  
they are too hasty and  
jump at conclusions,  
as you saw in the problem above.



So do not make a similar mistake  
again  
by coming to hasty conclusions about  
Democracy.

We are coming back to Democracy  
later.

In the meantime, please remember that  
you can fool  
"ALL of the people SOME of the time  
but NOT ALL the people ALL the time."

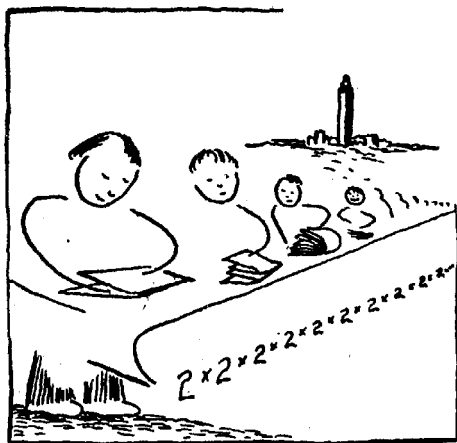
And since you are one of the people  
yourself,  
and don't want to be fooled  
if you can help it,  
you must be prepared to think straight.  
And, incidentally,  
don't fool yourself either  
by thinking that this can be done  
without any effort at all on your part.  
Perhaps this little book will help  
to smooth the road for you.

The Moral: Don't be a  
Conclusion-Jumper.



## II. DON'T HIT THE CEILING

Let us try another one,  
and this time give it a little more thought:  
Suppose you had a paper napkin,  
say about three-thousandths ( $.003$ )  
of an inch thick.  
Now lay another similar napkin on top of it;  
the two will of course be twice as thick as one,  
 $.003 \times 2 = .006$ ,  
or six-thousandths of an inch thick.  
Now put two more napkins on top of that  
making 4 in all,  
which are  $.003 \times 4 = .012$ ,  
or twelve-thousandths of an inch thick.  
Continue this process,  
each time doubling the number of napkins,  
thus:  
the first time you had 1 napkin,  
the second time you had 2,  
the third time, 4,  
the fourth time, 8,  
the fifth time, 16,  
and so on,



doubling the number each time,  
as we said before.

Now continue this 32 times.

The question is:

**HOW HIGH WILL THE PILE OF NAPKINS BE?**

Do you think it will be 1 foot high?

Or will it be as high as

a normal room, from floor to ceiling?

Or as high as the Empire State Building?

Or what?

The correct answer is not necessarily  
any of these.

What do YOU think?

Decide BEFORE you turn this page.

Let us again make out a table,  
showing clearly what was done:

	NO. OF NAPKINS	THICKNESS
1st time	1	.003 in.
2nd time	2	.006 in.
3rd time	4	.012 in.
4th time	8	.024 in.
5th time	16	.048 in.
6th time	32	.096 in.
7th time	64	.192 in.
8th time	128	.384 in.
9th time	256	.768 in.
10th time	512	1.536 in.
11th time	1024	3.072 in.
12th time	2048	6.144 in.
13th time	4096	12.288 in.
14th time	8192	24.576 in.
15th time	16384	49.152 in.
16th time	32768	98.304 in.
17th time	65536	196.608 in.
18th time	131072	393.216 in.
19th time	262144	786.432 in.
20th time	524288	1572.864 in.
21st time	1048576	3145.728 in.
22nd time	2097152	6291.456 in.
23rd time	4194304	12582.91 in.
24th time	8388608	25165.82 in.
25th time	16777216	50331.65 in.
26th time	33554432	100663.3 in.
27th time	67108864	201326.6 in.
28th time	134217728	402653.2 in.
29th time	268435456	805306.4 in.
30th time	536870912	1610612.7 in.
31st time	1073741824	3221225.5 in.
32nd time	2147483648	6442450.9 in.

In other words,  
the final pile of napkins is  
6,442,451 in. thick.  
To change this to feet,  
we must divide it by 12, obtaining:  
536,871 feet.  
Or perhaps you would like  
the answer in miles!  
In that case, divide now by 5280,  
since, as you know,  
there are 5280 feet in 1 mile.  
Thus we get:  
nearly 102 miles!  
Remember that 1 mile  
is about 20 city blocks.  
Now imagine a pile of napkins  
over 100 miles in height!

Are you surprised again?  
Did you get your answer by a "hunch"?  
Or did you try to do it experimentally  
by actually piling the napkins up?  
Or did you calculate it as we did?

Let us discuss these various methods  
a bit:

As regards a hunch,  
we wish to make two points very clear:

- (1) Some of our hunches are RIGHT  
and some of them are WRONG.  
The only way to tell  
which is which  
is to FOLLOW the hunch and  
check it up.
- (2) Scientists and mathematicians  
also have hunches—  
some of their best ideas  
have been hunches;  
but these do not become  
respectable Science and  
Mathematics  
until they are  
checked and double-checked.

This is one very essential difference  
between the behavior of T. C. and  
that of a scientist.

*T. C. is apt to think that*  
if he is good at hunches sometimes,  
he may rely on them always.

But the fact is that

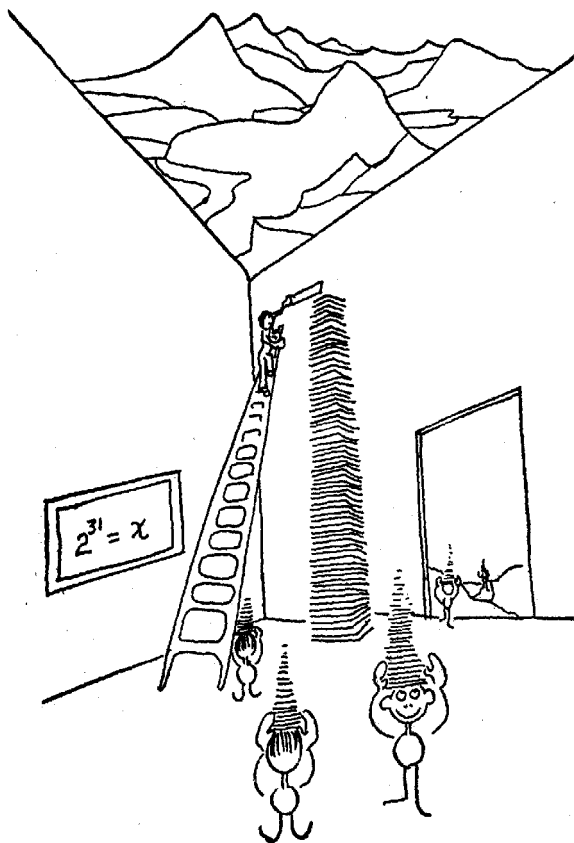
**EACH INDIVIDUAL HUNCH MUST BE  
CHECKED AND DOUBLE-CHECKED!**

Now as regards the experimental method:  
this is generally known as  
a very "practical" method:

"If you actually DO a thing,  
you cannot fail to get  
the right answer."

Often this is true,  
but you can easily see that  
in this particular problem  
it is scarcely practical to  
pile up napkins 100 miles high!  
You would surely  
hit the ceiling  
if you tried it!  
In short,  
do not be too sure of  
what is "practical" until  
you have examined  
the problem in question.

Finally,  
we have the method of calculation:  
this method was, as we saw,  
by far the best in this case.  
So let us NOT say that  
Mathematics is IMPRACTICAL  
whereas  
doing things with your hands is  
PRACTICAL.  
This is SOMETIMES true,  
but NOT ALWAYS!  
If you think the calculations  
were tedious,

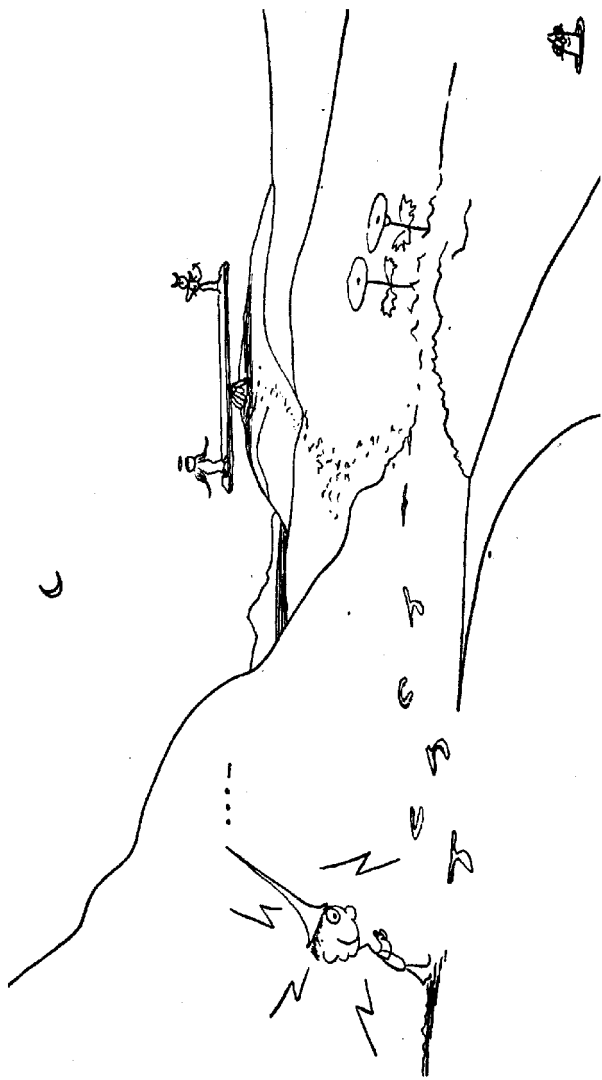


we must point out:

- (1) At least they were  
not as tedious as  
piling up the napkins  
would have been!
- (2) There is a much shorter way  
to calculate the answer—  
BUT  
for this you need to know  
a little MORE Mathematics:  
namely,  
a chapter in Mathematics  
known as Logarithms.  
We shall not explain it here,  
for it is already explained in  
any book on Algebra.  
You can look it up.  
And,  
with a little effort,  
thus learn a method which is  
useful on MANY occasions.

Please remember that  
it takes a little effort  
to drive a car,  
or to swim,  
or to do almost anything.  
But, if the result is worth while,





why growl at the effort?  
After all,  
the only way to make no effort at all  
is to be dead!

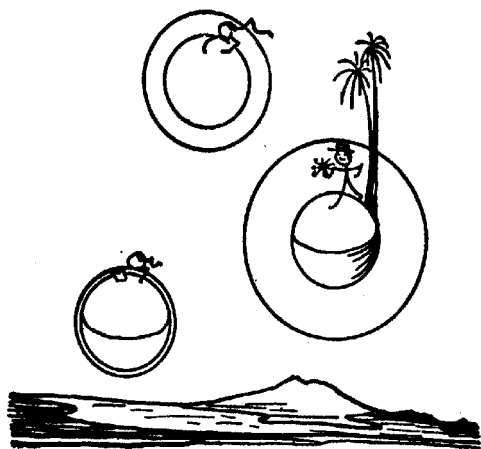
The Moral: Wake up and LIVE!  
And  
follow your hunches and  
check them!

### III. TISSUE-PAPER THINKING

Now that you are convinced that  
we must be careful and  
think more delicately,  
you are ready to tackle  
another question:

Suppose there were a steel band  
fitting tightly around  
the equator of the earth.  
Now suppose that you remove it  
and cut it at one place,  
then splice in an additional piece  
10 feet long,  
so that the new band is  
10 feet longer than the original one.  
If you now replace it on the equator,  
it would fit more loosely,  
would it not?

The question is:



How large a space would there now be  
between the band and the earth?

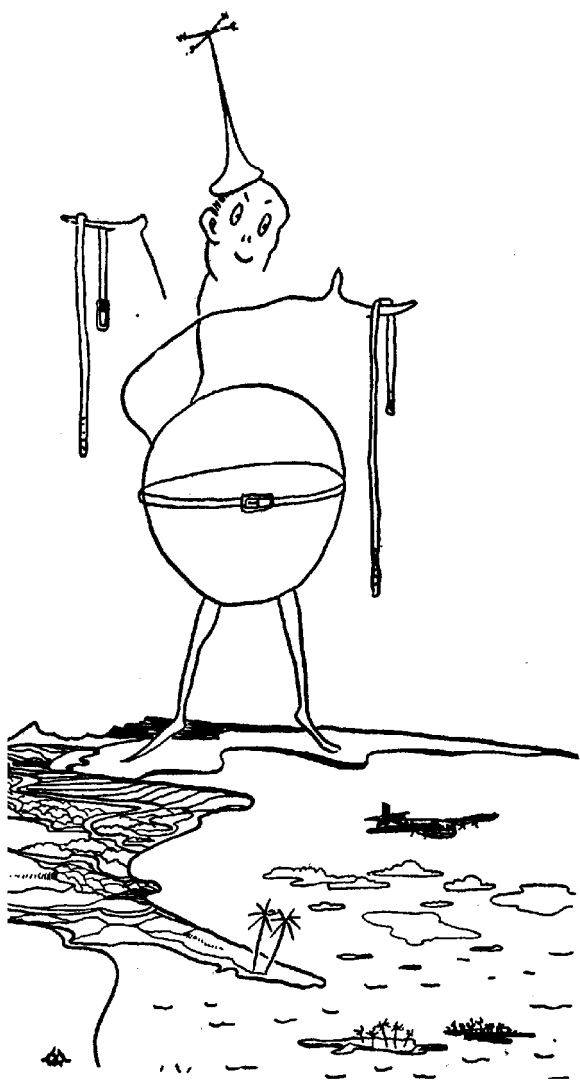
Would it be large enough for

- (a) a man, 6 feet tall, to walk through,
- (b) a man to crawl through  
on hands and knees,
- (c) a piece of tissue paper  
to just slip through?

**ANSWER BEFORE TURNING THE PAGE.**

Did you say (c) is the right answer?  
Perhaps this idea "flashed"  
into your mind  
because you felt that  
10 feet could not make  
much difference  
in a band which was  
thousands of miles long  
in the first place.  
Or perhaps you had learned  
not to trust a "flash" too readily,  
and decided to calculate  
the answer  
in the following way:  
"Since the distance around  
the equator is 25,000 miles,  
dividing 25,000 miles into 10 feet  
gives a very small amount,  
and therefore  
I still think that  
(c) is the right answer."

But such a manipulation of numbers  
can scarcely be called  
"Calculating the answer."  
For what justification is there  
for dividing these numbers?  
What is the THEORY behind this labor?  
On more careful consideration  
you must admit that



there is really no reason  
for doing this.

*In other words,*  
without a theory, a plan,  
the mere mechanical manipulation  
of the numbers in a problem  
does not necessarily make sense  
just because you are  
using Arithmetic!

Now let us really examine  
this problem  
sensibly:

You probably know that  
the circumference (C) of any circle  
may be found by  
multiplying its radius (R) by  $2\pi$ ,  
where  $\pi$  is a Greek letter  
(pronounced "pie")  
and is a symbol whose  
approximate value is 3 and  $1/7$ .  
Expressing this fact about a circle  
more briefly,

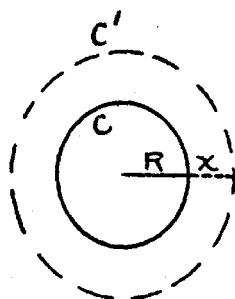
we may say

$$C = 2\pi R.$$

And this is true of ANY circle,  
no matter how large or how small.

Now if we increase the radius  
by an amount  $x$  (see Fig. 1),

and make a new, larger, circle  
whose radius is now  $R + x$



the new circumference would now be

$$C' = 2\pi(R + x)$$

would it not?

This may be written

$$C' = 2\pi R + 2\pi x.$$

If we now compare this with  
the value of  $C$  given above,  
namely,

$$C = 2\pi R$$

we see that

$C'$  is more than  $2\pi R$

by an amount  $2\pi x$ .

In other words,

Just as  $5(2 + 7)$  may be written  
 $5 \times 2 + 5 \times 7$ , since in either case  
the answer is 45.



increasing the radius by  $x$   
increases the circumference by  $2\pi x$   
or by  $6\frac{2}{7}$  times  $x$ .

Now, then,  
if we increase the circumference  
by 10 feet,  
as in our problem,  
we have

$$6\frac{2}{7}x = 10 \text{ feet,}$$

and consequently

$$x = 10 \div 6\frac{2}{7}$$

or

$$x = \text{about } 1\frac{1}{2} \text{ feet.}$$

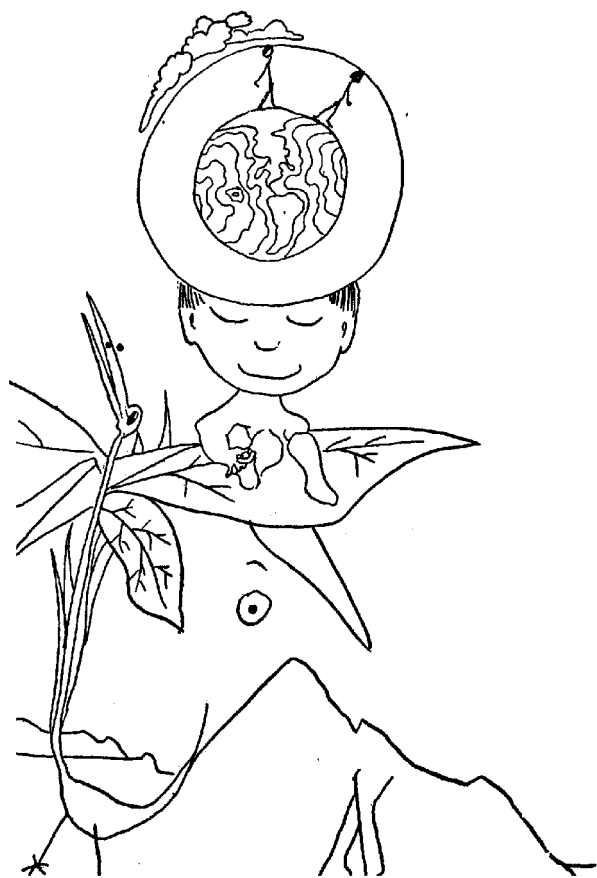
That is to say,  
an increase in the circumference  
of 10 feet  
results in an increase in the radius  
of about 1 and  $\frac{1}{2}$  feet,  
and consequently

(b) on page 33 is  
the correct answer to our problem.

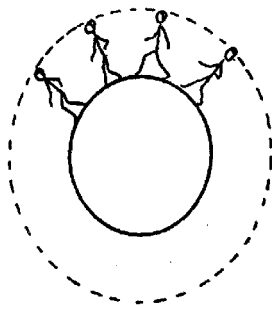
So you see,  
you must not calculate mechanically,  
like a robot.

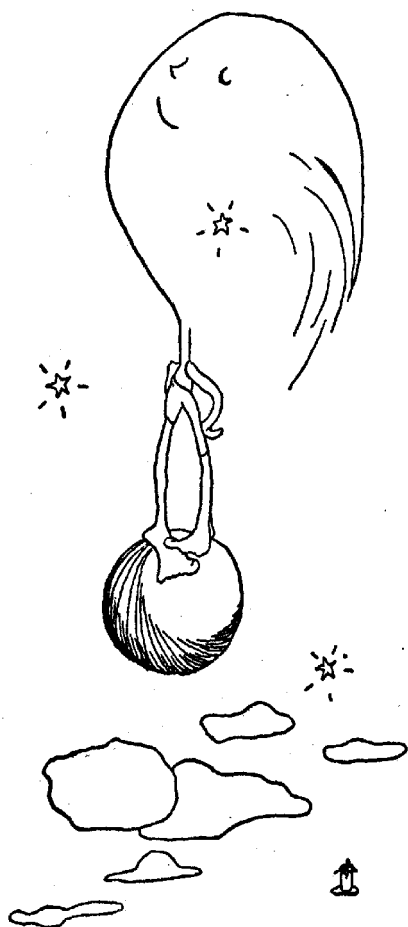
Now that you have seen  
a sensible method of solving  
this type of problem,  
try this one:

Suppose that you went  
on a long walking tour



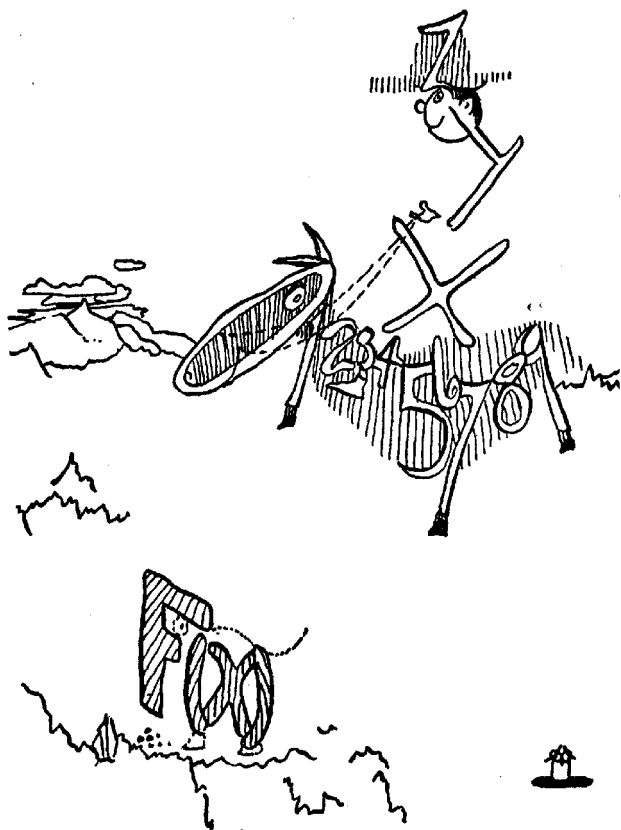
around the equator  
(assume that the earth is  
a perfect sphere),  
and suppose that you are 6 feet tall,  
how much further would  
your head go  
than your feet?!  
Perhaps you are surprised  
at the very idea  
that when you go for a walk  
your head CAN go further  
than your feet!  
Perhaps you think that  
this is against "Common Sense."  
But a careful look at Fig. 2  
will doubtless convince you  
that this idea is entirely sensible!  
For, as your feet travel  
along the inner circle,  
your head obviously goes along  
the outer, dotted, circle.





The Moral: Your head CAN go further  
than your feet!

ΔΧ



#### IV. GENERALIZATION

No doubt you are aware of the fact  
that the formula about the circle  
in Chapter III is  
"Algebra."

And perhaps you can guess from this  
that one way in which  
Algebra is different from Arithmetic  
is:

Whereas in Arithmetic we do  
one specific problem at a time,  
in Algebra we give  
a GENERAL rule for doing many  
problems of a certain type.  
Thus when we find  
the area of a rectangle  
whose base is 4 in. and altitude 2 in.  
and get 8 square in.,  
THIS IS ARITHMETIC.

But when we write  
(1)  $A = ab$   
which says that  
to find the area,  $A$ ,  
of ANY rectangle,  
we must multiply the altitude,  $a$ ,  
by the base,  $b$ ,  
THIS IS ALGEBRA.

In other words,  
Algebra is more GENERAL  
than Arithmetic.  
But perhaps you will say that  
this is not much of a difference—  
since in Arithmetic  
we also have general rules,  
but they are given in WORDS,  
instead of in LETTERS as in (1).  
Thus in Arithmetic we would say:  
“To find the area of any rectangle  
multiply its altitude by its base,”  
whereas in Algebra we say:

$$A = ab,$$

but, after all, you may feel that  
this is merely a matter of  
a convenient shorthand  
rather than anything radically new.

Now the fact is that  
it is not merely a question of

a convenient shorthand,  
but  
by writing formulas in this  
very convenient symbolism—  
especially when a formula is  
much more complicated than  
the one given above—  
we are able to tell  
**AT A GLANCE**  
many interesting facts  
which would be very difficult to  
dig out from a complicated  
statement in words.  
And, furthermore,  
when we learn to handle  
the formulas,  
we find that  
we are able to solve problems  
almost automatically  
which would otherwise require  
a great deal of hard thinking.  
Just as,  
when we learn to drive a car  
we are able to “go places”  
easily and pleasantly  
instead of walking to them  
with a great deal of effort.  
And so you will see that  
the more Mathematics we know  
the **EASIER** life becomes,



for it is a TOOL with which  
we can accomplish things  
that we could not do at all  
with our bare hands.  
Thus Mathematics helps  
*our brains and hands and feet,*  
and can make  
a race of supermen out of us.

Perhaps you will say:  
"But I like to walk,  
I don't want to ride all the time.  
And I like to talk,  
I don't want to use  
abstract symbols all the time."  
To which the answer is:  
By all means enjoy yourself by  
walking and talking,  
but when you have a hard job to do,  
be sure to avail yourself  
of all possible tools,  
for otherwise  
you may find it impossible  
to do it at all.

And so,  
if you wish to be an engineer  
and build bridges and things,  
you must know Mathematics.  
If you wish to figure out

how much money to put away now  
so you may have  
a comfortable income in your old age,  
use a formula.

If you want to know  
how much interest  
you are REALLY paying  
when you borrow money or  
when you buy things on installments,  
use a formula.

And, mind you, these are  
algebraic formulas:  
some of these problems  
CANNOT be solved by using  
Arithmetic alone!

••

You would be surprised to know  
about some of the  
remarkable and useful formulas  
that would help you  
if you would make  
a little effort to  
find out about them.

In fact  
the trouble with the world today  
is not that  
we have too much Mathematics,  
but that we do not yet have enough.  
For, there are as yet

no powerful Mathematical methods in  
Psychology,\*  
the Social Sciences,  
and other important domains.  
So that even the best workers  
in those fields are,  
figuratively speaking,  
still using their bare hands,  
and walking (rather than gliding),  
and talking in ordinary language.  
Perhaps, in these domains,  
we ARE having fun all right,  
but  
we are getting nowhere  
very fast,  
for our wars are steadily becoming  
bigger and better.

No doubt someone will say:  
"But the war-makers  
DO use modern machinery which  
IS based on Mathematics.  
Science is really to blame for  
the success of Hitler,  
and therefore

\* But see the work now being done,  
described in

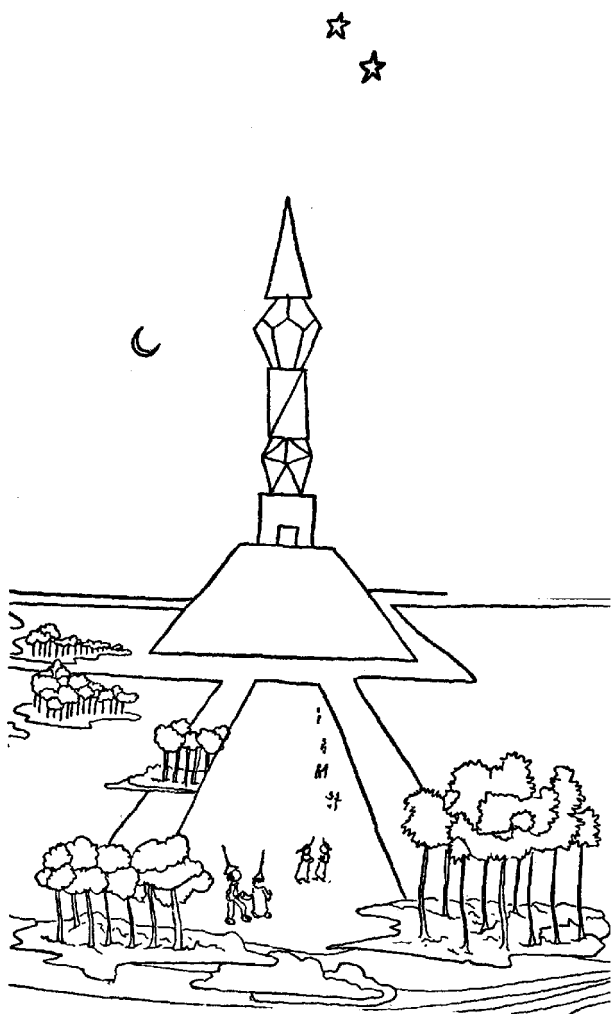
- (1) "Mathematico-deductive Theory of  
Rote Learning" (Yale University Press).
- (2) L. L. Thurstone: "The Vectors of Mind"  
(University of Chicago Science Series).

it cannot possibly guide us to  
the good life."

Now we hope to show here  
that this is not so —  
that Science and Mathematics can  
not only protect us from  
floods and lightning and disease  
and other such physical dangers,  
but have within them  
a PHILOSOPHY which  
can protect us from  
the errors of our own  
loose thinking.

And thus they can be  
a veritable defense against  
ALL evil—  
a Totem Pole—  
if we would but examine into them  
carefully.

The Moral: Streamline your mind  
with  
Mathematics.



## V. OUR TOTEM POLE

Let us symbolize our Totem Pole  
by a column made up of  
the five well-known regular solids,  
as shown in the  
drawing on the opposite page.  
And let us think of the solids as  
separate rooms, in each of which  
a certain aspect of Science  
is presented.

We shall now take you  
on a guided tour of these rooms.

The First Floor, the CUBE,  
contains all the scientific gadgets  
with which we are all so familiar:  
automobiles and refrigerators  
and radios and airplanes,

and what seems like googols \* of others.  
This is of course also the room  
in which you will find  
tanks and bombers and  
all the paraphernalia of war.  
And this is why some people say  
that Science is amoral,  
since it produces,  
with equal indifference,  
the peaceful toys we enjoy so much  
as well as the instruments of  
destruction.  
But these people have probably  
never climbed the magic staircase

\* A "googol" is a very large number,  
namely,  $10^{100}$ , or 1 with a  
hundred zeros after it.  
The term "googol" was  
invented in fun by  
a nephew of one of our  
great American mathematicians,  
Professor Edward Kasner of  
Columbia University.  
It is becoming a popular word  
and will doubtless soon be  
in the dictionary.  
If you wish to know more about googol  
and "googol plex,"  
and many other interesting things,  
see  
"Mathematics and the Imagination"  
by  
Kasner and Newman (Simon and Schuster).

which leads from the gadget room  
up into the other rooms of  
our Totem Pole,  
and are entirely unfamiliar with  
their contents.

Let us therefore go up to  
the Second Floor, the ICOSAHEDRON.  
Here we find  
a great industrial laboratory—  
this is where the gadgets are  
invented, tried out, manufactured;  
the men working on this level  
are not advertising and selling,  
they are inventing.  
They are told by  
the people who hire them:  
“We want a brighter light,  
a cheaper light,  
a more smoothly running car,  
an effective defroster for  
airplanes”—  
and a thousand and one other things.  
These research men do not  
let their minds roam around  
looking for interesting problems.  
Their problems are handed to them,  
and they must solve them within  
a very reasonable length of time,  
“or else.”



Only "practical" men are  
wanted here,  
and not oversentimental ones.  
For they may be told at any moment  
to find effective ways of  
killing people—  
they must make  
the best long-range guns,  
the best poison gases,  
the bombs which can  
destroy the most people.  
But you thought we promised  
to get away from all this  
as we climbed upward:  
and yet this floor seems to have  
even more diabolical possibilities  
than the first one.  
Perhaps if this second floor  
were destroyed,  
the war paraphernalia of  
the first floor might  
become obsolete and die out  
of its own accord.  
Are not these scientific inventors  
the real devils after all?

But let us climb another flight  
and see what goes on in  
the OCTAHEDRON.  
These men are doing research in

"Pure" Science.

They are not employed by  
manufacturers or governments;  
they are usually  
professors at universities who  
select their own problems because  
they are interested in them.

They are not concerned with  
any practical applications of  
their ideas.

They are the theoretical men—  
they ask the most "useless" questions.

For instance:

"What happens when you mix  
sugar and water and lemon?"

They call it "Sugar Hydrolysis"  
instead of "Lemonade."

They study it in different solutions,  
carefully varying the  
relative amounts of  
the substances involved,  
and examine them with a polariscope  
for days and days, and years and years,  
keeping careful records  
and publishing the results in  
scientific journals.

Will these investigations  
make them rich?  
Or fat?

Or benefit them in any "practical" way?

Not at all.

Then why do they do it?

The answer is that  
they are just driven by  
Curiosity.

Once in a while they are consulted by  
the men on the second floor,  
but not so very often.

Usually they just  
write up their results and  
die without knowing whether  
these will ever have any  
practical use.

But the fact is that  
their results ARE very often,  
in the long run,  
used by some second-floor scientist.

Indeed,  
these second-story men find that  
they must study the work of  
the "pure" scientists  
constantly.

But usually it is the work of  
the "pure" ones of the past—  
work which has already found its way  
into the textbooks which  
they have studied at the  
institutes of technology,  
rather than the current work

published in the journals of  
Pure Science.

In fact,  
the gentlemen working at  
any given time on the  
second and third floors  
seem to have very little in common:  
the second-floor men consider  
the upper-floor men to be  
"wild-eyed, absent-minded  
college professors.

Some of them are perhaps just  
crackpots,  
who knows?

It is safer to go to the theory of  
the established past,  
which has been duly  
tried and tested."

And, on the other hand,  
the inhabitants of the third floor  
look down upon the second-story men,  
considering them to be mere  
"hirelings and ignoramuses,"  
and prefer to leave their results to  
the second-story men of the FUTURE,  
"who will be in a better position to  
appreciate them."

But, granting even that  
this will be so,

what guarantee have we that  
the uses that their ideas will find  
WILL be decent, moral uses?  
How do we know that  
they are not storing up just  
a lot of additional trouble for the  
unfortunate future generations?  
No,  
let us climb up further,  
and look at the Fourth Floor,  
the DODECAHEDRON.

Here we find  
the Mathematicians—  
not the “Pure” Mathematicians,  
for they live on the Fifth Floor,  
in the TETRAHEDRON garret,  
with the Modern Artists.  
The fourth-floor mathematicians are  
the ones who know the  
Classical Mathematics of the past  
and apply it to  
the scientific findings of the  
“Pure” Scientists of the third floor.  
They take the scientific data  
and organize it  
and study it with  
all the mathematical machinery  
at their command.  
If a second-story man ever

happens in on the fourth floor,  
which is very rare,  
he can hardly control his laughter.  
These men seem to him to be  
even more wild-eyed than those  
on the floor below,  
but the guide tells him:  
"You ain't seen nothin' yet,  
wait till you see the Top Floor,  
the TETRAHEDRON."  
At least on this fourth floor  
you hear them mention  
Geometry and Algebra and Calculus,  
subjects you have heard about  
in high school or in college.  
But on that top floor,  
they draw geometric figures on  
doughnuts and pretzels  
(no fooling!)  
and on rubber sheets.  
And they have up there  
Algebras and Arithmetics in which  
twice two is NOT four!  
In which  $3 + 2$  does NOT give  
the same answer as  $2 + 3$ ,  
nor is  $5 \times 6$  equal to  $6 \times 5$ !!  
They are indeed fit companions for  
the Modern Artists who  
share the garret with them!  
They are lucky if they can even

get a job!  
And yet the connoisseurs say that  
their work is  
tremendously important for  
the future.  
Indeed,  
if you trace back some of the  
most practical and useful gadgets,  
you will find that  
if it had not been for a series of  
"wild-eyed," "impractical" men,  
these gadgets could not exist today.

As you will see in the next chapter.

## VI. THE TOTEM POLE (Cont.)

Take the radio for example,  
with all its variety of  
concerts and important  
broadcasts of all kinds.  
Trace it to the second floor  
and you will find that  
many men on that floor have been  
*improving reception by inventing*  
*better tubes and aerials, etc.*  
But all this could not have happened  
had it not been for  
a man named Marconi,  
a second-story man,  
who sent the first  
crude radio messages.  
And even his work  
would have been impossible  
had it not been for  
another man, named Hertz,  
who worked on the third floor,



and who proved that the very idea  
of sending a wireless message  
was actually possible,  
since he demonstrated the existence of  
electromagnetic waves.

But where did he get the idea of  
even looking for these waves?

Why, from a fourth-floor man,  
of course,

a man named Clerk Maxwell,  
who first conceived the idea of waves in  
an "electromagnetic field" and  
applied the Calculus to it, obtaining  
a set of differential equations  
from which he declared  
the consequence followed that  
there MUST be  
electromagnetic waves.

And, as we have already said,  
Hertz subsequently proved  
that he was right.

And, obviously,  
Maxwell could not have done his job  
had not Newton invented the Calculus.

And so it goes.

Take any gadget you like  
and trace it back,  
and you will find that

invariably you will have to  
go up into  
all the five floors  
before you can have its complete story.

"But," you will say,  
"you have not proved  
your initial point  
at all,  
since the same is true of  
tanks and bombers also,  
and therefore  
Science IS indifferent to  
Good and Evil,  
and IS amoral after all."

You will soon admit, however,  
if you read again  
the story we have told you,  
more attentively this time,  
that Science is trying to  
tell you something else,  
if you will but listen.  
For instance,  
go back and you will see that  
in the little story of the radio,  
there are  
Americans,  
Italians,  
Germans,

Englishmen.

If you take the airplane,  
you will also find  
Russians,  
Frenchmen,  
and others.

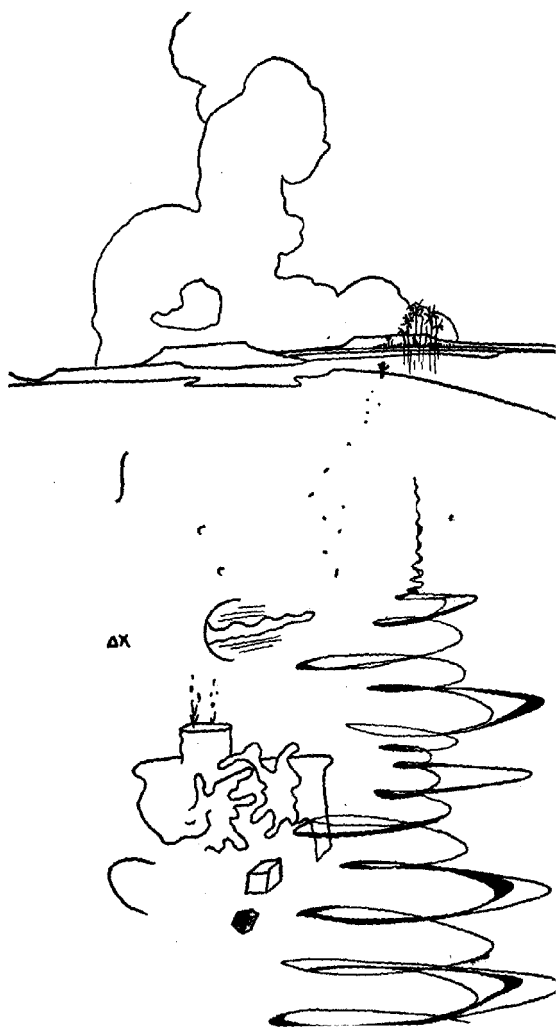
In short you will be very much  
impressed by the fact that  
**SCIENCE IS INTERNATIONAL**,  
that it is trying to tell us that  
Hitler's racial theories are  
utterly false.

It is also trying to tell us—  
if we would only listen—  
that co-operation is essential  
for accomplishing things,  
that it is really absurd  
for the first- and second-story men  
to laugh at those who live upstairs,  
or for the latter  
to look down upon the others.  
For they are all needed  
to do the job.  
Is not this **DEMOCRACY**?

Thus we see that  
Science is **NOT AMORAL**,  
but has a **PHILOSOPHY**  
to offer us,  
provided that we do not  
merely identify Science with

first-floor gadgets,  
and thus  
cut off its HEAD  
(the upper floors!)  
and stop its  
BLOOD STREAM  
(the interrelationship  
between ALL the floors)!

And as we tell you more about  
those strange  
algebras and geometries we mentioned,  
you will see that  
Mathematics has many important  
messages for us—  
that it is trying to tell us that  
VARIOUS mathematical systems  
are possible,  
that they are all man-made  
and controllable by man,  
and that  
if you apply this idea  
to the social world,  
you will realize that  
it is up to you  
to build a good world if you want one—  
that man has a great deal more  
freedom and creative ability than  
he is sometimes aware of.  
The idea of a fixed “human nature”  
that has us by the throat



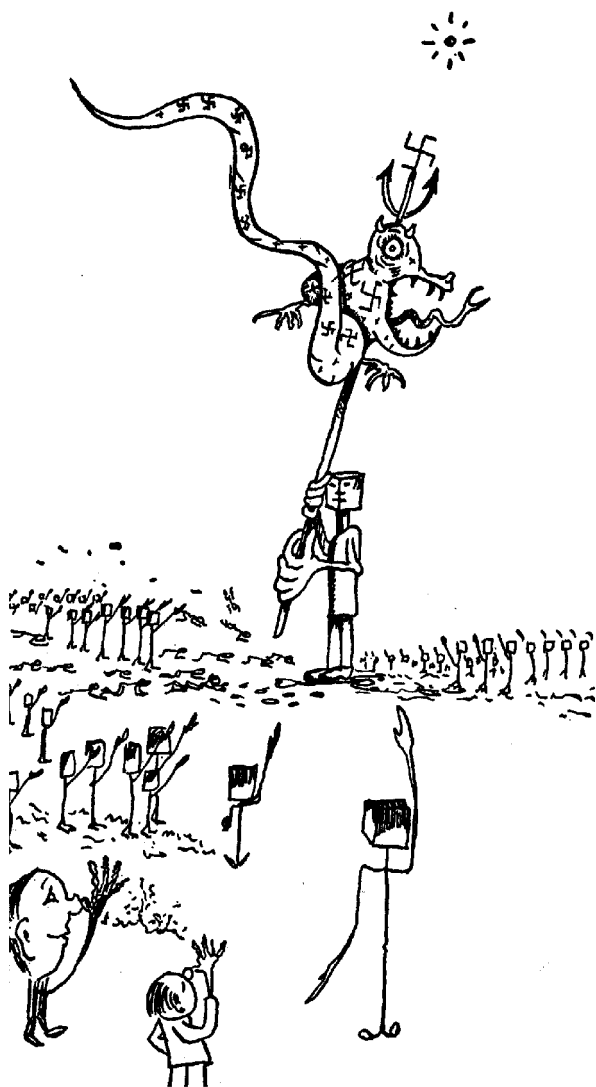
is just a fiction,  
for  
the activities on the top floor  
are trying to tell us that  
**HUMAN NATURE HAS  
INFINITE POSSIBILITIES.**

In short,  
it is not the guns and tanks  
which are the real evils—  
for a gun may be a great “good”  
under certain circumstances.  
But rather  
such false IDEAS as  
“Nationalism,”  
“Dictatorship,”  
narrow Views of  
“Human Nature,”  
etc.,  
are the real DEVILS.  
Thus  
**FALSE IDEAS ARE MORE DANGEROUS  
THAN GUNS!!**  
Guns and tanks are mere tools,  
they may be used for good or evil.  
But they are only  
first-floor gadgets,  
whereas the philosophy of science,  
which comes from  
a contemplation of all the floors,  
and of their relationship

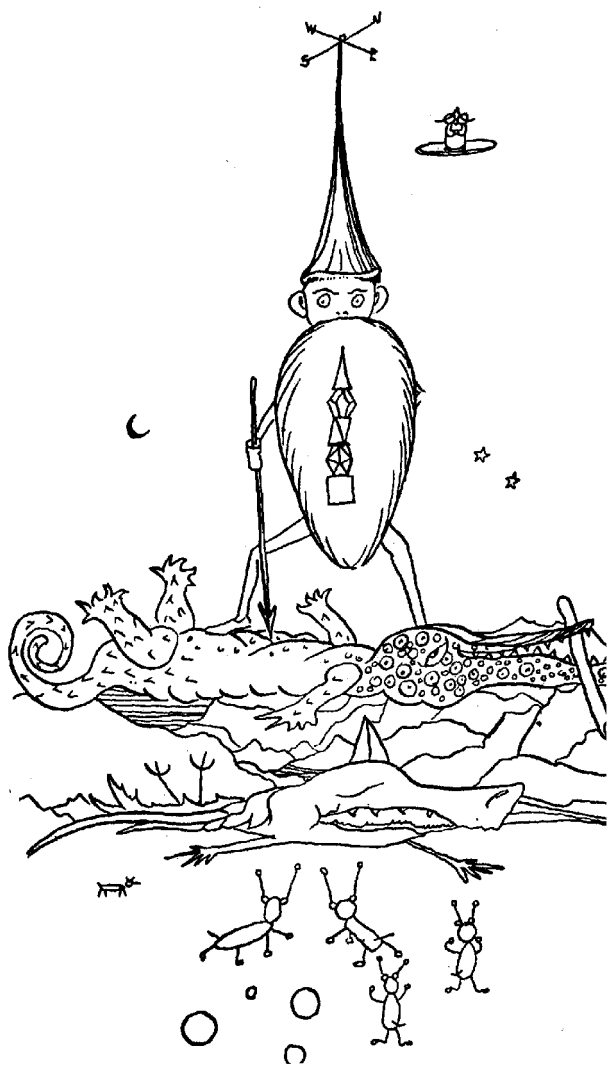
to each other,  
has for us  
unmistakable messages:  
we must rise up above  
the first and second floors  
and realize that  
these alone are  
**NOT SUFFICIENT**  
for the human race,  
whose nature is so  
beautifully revealed by  
a study of Science as a whole,  
which, as we have seen, has  
Internationalism and Democracy  
at its very heart.

We therefore advocate:

- (1) A broader view of Science which  
enables us to appreciate the  
philosophy in it—  
Science as a whole,  
as seen in the Totem Pole,  
can really protect us from evil.
- (2) A more appreciative attitude  
toward the top-floor men.  
By knowing how much we owe  
to the top-floor men of the past,  
we should stop treating them with  
the brutality with which







they have been treated  
in the past—  
just because they do not use  
their energy to make themselves  
physically comfortable.  
And we should stop heckling them  
by asking them:

“What is the practical use of  
what you are doing?”

or

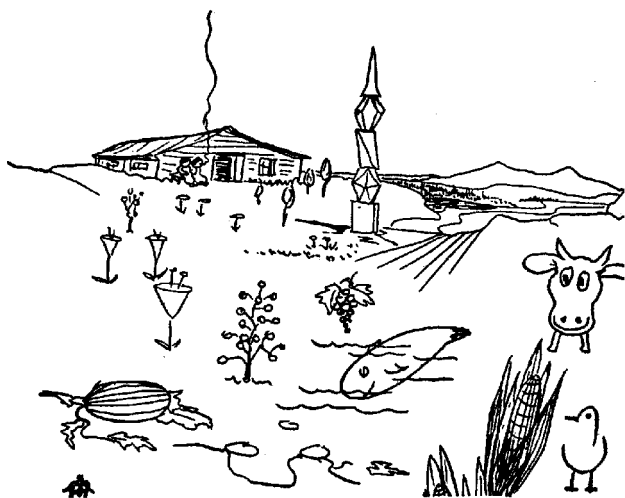
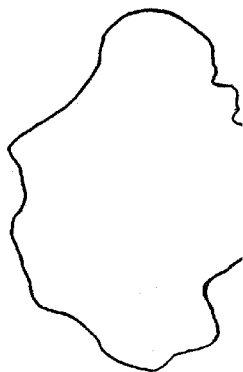
“What does this mean for  
The Average Man?”

Since the truth is that  
they themselves do not know.

Their work is as much  
a “Natural Phenomenon” as  
a natural oil well or  
natural gas or  
mountains or  
rivers.

Let us give them the  
freedom they need  
to do what is in them to do.  
Let us turn their garret into a  
penthouse,  
and marvel at their  
strange products.

Perhaps some day we shall find  
a “practical” use for them,



as has so often happened  
in the past.  
And besides,  
the philosophical implications  
of their work  
already make them  
invaluable to us NOW--  
as we shall see.

The Moral: Oh, listen to the  
Totem Pole!

## VII. ABSTRACTION

You saw in Chapter IV that  
GENERALIZATION is  
one of the principal advantages  
that Algebra has over Arithmetic.  
In fact GENERALIZATION is  
one of the fundamental methods  
of obtaining new results  
in all of Mathematics.

Perhaps someone will say:  
"But generalization is not the  
private property of mathematicians,  
every man knows that  
all women are silly.  
Every woman knows that  
all men are fools.  
And everyone knows that  
all Jews are  
bankers AND Communists."  
We need hardly say  
that these generalizations are  
NOT VALID,

whereas in Mathematics  
we take pride in making  
our generalizations with  
MUCH GREATER CARE.

Now, in Geometry, as you know,  
we deal with  
the relationships between  
points, lines, planes, and so on,  
and study the properties of  
various figures  
(triangles, circles, etc.)  
and  
various solids  
(prisms, spheres, etc.) .  
And, as you also know,  
we draw diagrams of plane figures  
on a blackboard or a piece of paper,  
and make models of  
three-dimensional objects,  
to help us visualize  
the things we are discussing.  
But of course you realize that  
a point drawn on a blackboard  
with chalk,  
or on a piece of paper  
with even the finest pencil or pen,  
is much too large for  
a mathematical point,  
which is supposed to have

no dimensions at all—  
no length, no breadth, no thickness.  
And, similarly,  
a circle drawn with  
even the best instruments  
is only a crude representation  
of a mathematical circle.  
Thus  
the things with which we deal in  
Geometry  
are ABSTRACTIONS of actual things  
in the physical world.  
And just because they ARE  
abstractions,  
they are therefore  
EXACT instead of APPROXIMATE.  
For example,  
every point on the circumference of  
a mathematical circle  
is at EXACTLY the same distance  
from the center.  
But you might say:  
“Even if they are exact,  
what good are they when  
they exist only in the mind?”  
You will soon see  
what a mathematician can do  
with abstractions and  
how they can be applied  
to the actual world.

In fact,  
this power to ABSTRACT is  
one of the outstanding characteristics  
of human beings as  
compared with other animals.  
And this power is used not only  
by mathematicians,  
but also by  
artists, musicians, poets,  
and all other "human" beings.  
Perhaps some day  
we shall measure  
a person's "human-ness" by  
his power to abstract  
rather than by the I.Q.  
For a person who can  
be loyal to such  
abstract concepts as  
truth, justice, freedom, reason,  
rather than to  
an individual or a place,  
has the loyalty of a human being  
rather than that of a dog.  
Please do not think that  
we are using the word "dog"  
in a disparaging sense,  
for they are very dear animals.  
(Remember that you must not be  
a Conclusion-Jumper!)

But still they are animals and



not human beings.

But what are

"Truth,"

"Justice,"

"Freedom,"

"Reason,"

etc.?

Do these words really mean anything?

And how can we be loyal to them

if their meaning is not clear?

Are they not just "fakes,"

invented so that

some people can make slaves of others

by fooling them with such

meaningless abstractions?

Now you will see,

when you have finished this

little book,

that these concepts

"Truth," "Freedom," "Reason," etc.,

will become much clearer when

we examine into what is meant by

"Mathematical Truth,"

what kind of "Freedom" we have

in Mathematics,

what is considered good "Reason"

in Mathematics,

and so on.



You will see that  
*as mathematicians have been*  
*gradually forced to consider*  
*the fundamentals of Mathematics,*  
they have been obliged  
to consider the very nature  
of human thinking—  
both its powers and  
its limitations.  
For instance,  
what is the nature of a “proof”  
by human beings  
for human beings?

And, of course,  
this has a definite bearing on:  
“What are we humans anyway?  
What is the best that  
we can expect of ourselves?”

The Moral: Be a man—not a mouse.

## VIII. "DEFINE YOUR TERMS"

So far then  
we have said that  
GENERALIZATION and ABSTRACTION  
are very fundamental and useful  
human concepts.

And we must emphasize the fact that  
Mathematics is not the only domain  
in which these concepts are used.

For example,  
a great symphony  
does not have specific words  
like a popular song,  
and thus it abstracts an emotion  
rather than giving  
a particular instance of  
an emotion,  
and therefore has  
a wider application.

Similarly,  
a great portrait  
is more abstract than a photograph

because it does not represent  
the person as he looks at a  
particular moment,  
but abstracts what the artist  
considers to be  
the essential character of  
his subject.

Perhaps someone will say:

"I agree that this kind of  
abstraction

is good,

for I admit that a great portrait

has a wider scope

than a photograph.

But what about these MODERNS

who abstract to a degree

where the subject is no longer

recognizable at all?"

But let us not discuss

the MODERNS here,

for remember that

this Part I is called

"The Old," not "The New."

We shall discuss the MODERNS

in Part II.

For the present

we merely wish to point out

that

the concepts of

GENERALIZATION and ABSTRACTION

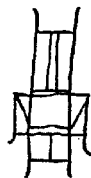
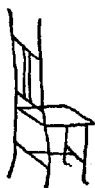
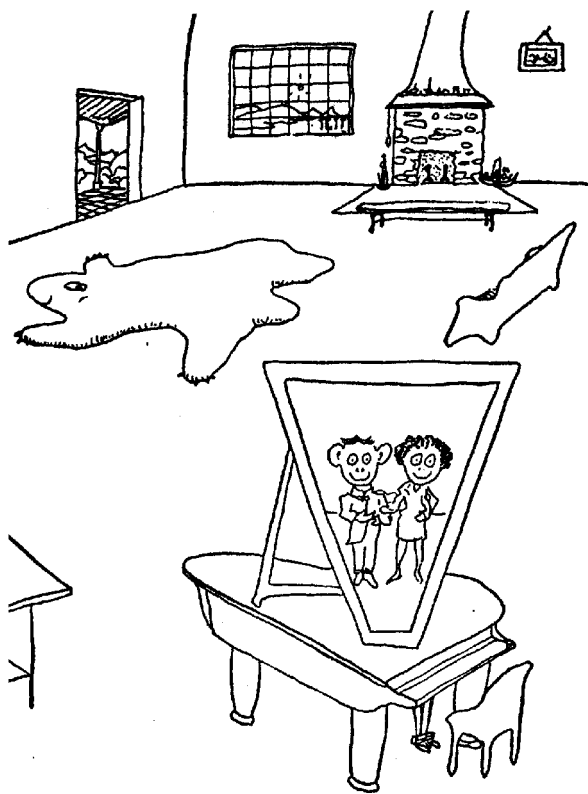
in Mathematics,  
as well as in Art, etc.,  
have an "OLD" and a "NEW" aspect.  
And the uses of them described above  
belong to the "OLD" in Mathematics  
just as portrait painting is  
an "OLD" form of abstraction  
in painting.  
And the "NEW" in Mathematics,  
as well as in Art,  
may also sound bizarre  
to the uninitiated.  
For example,  
as we have said before,  
to a modern mathematician  
 $2 \times 2$  does not have to be 4!!  
But do not let this frighten you,  
for when you have read Part II,  
you will have become  
so broad-minded  
(we hope)  
that such modern ideas  
will seem just as reasonable  
as anything you believe today.

But let us not anticipate;  
and continue with our story.

What other fundamental ideas  
do we find in Mathematics?

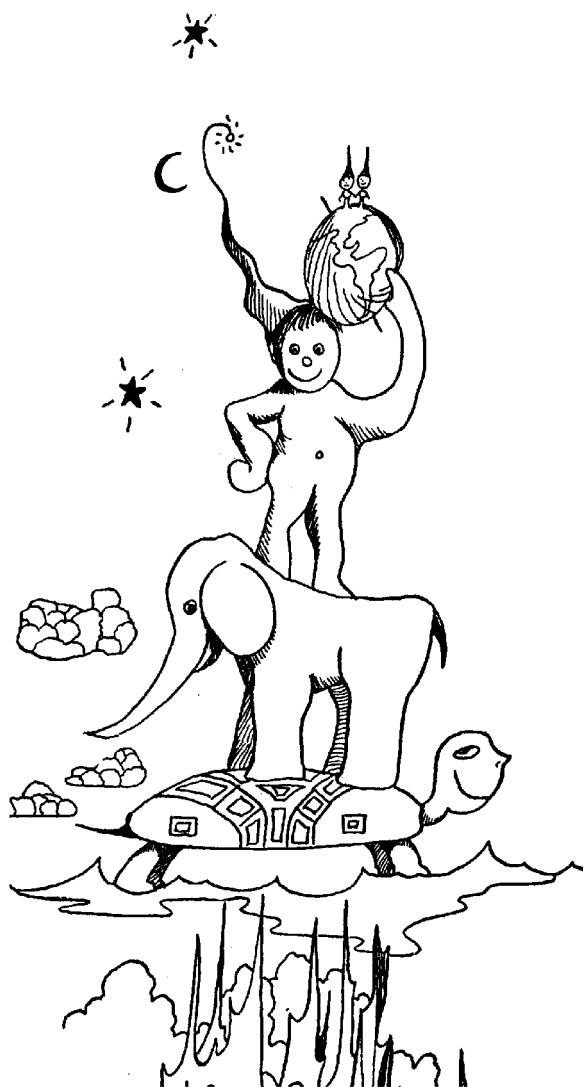
No doubt many of you will say:  
"Surely you will discuss the fact  
that Mathematics is a domain  
in which  
we prove everything,  
in which we carefully  
define all our terms,  
so that we know what  
we are talking about.  
And the moral of this  
will doubtless be that  
we should learn  
to define all our terms  
in ANY argument,  
and thus use  
the mathematical method  
as a model."

Well,  
we are sorry to disappoint you,  
but we must tell you that  
even Euclid,  
as far back as 300 B.C.,  
already realized that  
it is IMPOSSIBLE to  
define all of our terms or  
to prove everything,  
even in Mathematics!  
For, you see,  
since in a proof





every claim must be supported  
by something which has  
already been previously proved,  
and every term must be defined  
by something which has  
already been previously defined,  
obviously then  
at the very beginning of  
any system of thought  
we do not yet have  
anything to build on  
and therefore  
we must START with  
UNDEFINED terms and  
UNPROVED propositions.  
“But,” you will say,  
“it is not as bad as it sounds,  
because we can always begin with  
self-evident truths.”  
This is precisely what  
Euclid thought he did.  
And it was quite natural in  
those days  
for him to think so.  
But you will see in Part II that  
this is NOT the MODERN thing to do  
at all!  
However, let us at this moment  
continue with Euclid.  
He gathered together the





geometric knowledge of his time,  
and arranged it  
not just in a hodge-podge manner,  
but, as we said above,  
he started with what he thought were  
self-evident truths  
and then proceeded to  
PROVE all the rest by  
LOGIC.

A splendid idea, as you will admit.  
And his system has served  
as a model  
ever since.

But, as we promised above,  
you will see in Part II  
the very fundamental changes  
which mathematicians have been  
obliged to make in  
Euclid's system.

And, therefore,  
with all due respect to Euclid,  
we must not slavishly follow him  
TODAY,  
as so many of our Geometry texts  
do!

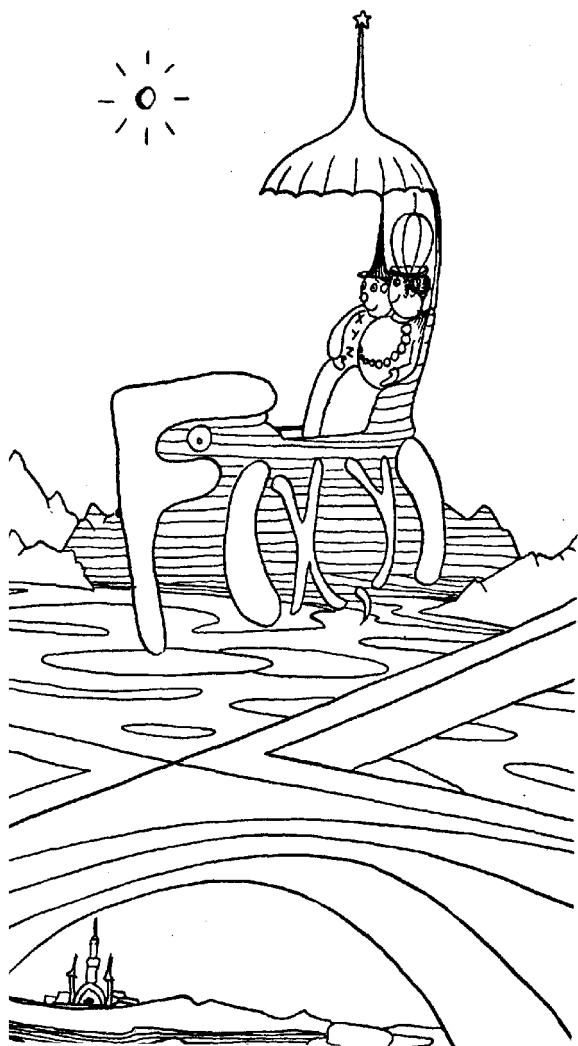
The Moral: Progress is made by  
respecting tradition  
without slavishly  
following it  
100 per cent!

## IX. A WEDDING

If you look back over the history  
of the human race,\*  
you will find that  
many useful things from  
Arithmetic and Algebra  
were known as far back as 4000 B.C.;  
that Geometry reached  
a high stage of development in  
the work of Euclid, about 300 B.C.  
Since then  
many more things have happened  
in Mathematics:

- (1) Algebra and Geometry have both  
been developed further.
- (2) They have been COMBINED into  
a new branch of Mathematics  
known as Analytic Geometry—  
by Descartes in the 17th century.
- (3) Many new Algebras and

In this connection read:  
"The Development of Mathematics" by  
E. T. Bell (McGraw-Hill).





- many new Geometries  
have been developed.
- (4) The FUNDAMENTAL IDEAS of  
all Mathematics  
have been carefully examined.
- (5) Logic has been inspected  
and new logics have arisen.
- (6) New applications of Mathematics  
to the study of the universe  
have been made.
- (7) And, as a result of all this,  
mathematicians have become  
much wiser,  
much more sophisticated.  
And their "common sense" has  
become so enlightened that  
they cannot help but look upon  
the more usual common sense  
of T.C. Mits as  
an adult looks upon the  
common sense of a young child  
who thinks that  
every man is his daddy.

Of course it takes a great deal of  
powerful thinking to become  
a great mathematician,  
but we believe it is possible  
to give T.C. a glimpse  
into the results,



without asking him to become  
a mathematician himself.

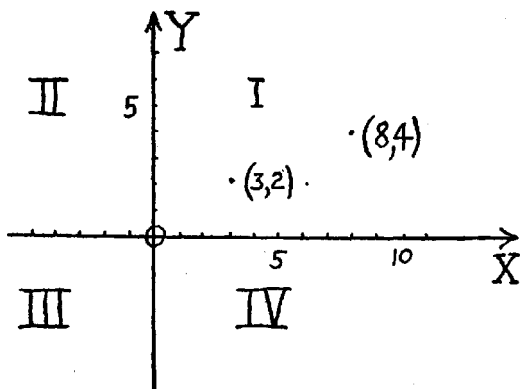
In this chapter we want to tell him  
about the wonderful

17th century wedding  
mentioned in (2) above—  
and about the offspring.

Descartes conceived the idea of  
associating Algebra and Geometry  
in the following manner:

If we draw two perpendicular lines,  
X and Y,

as shown in the next figure,  
thus dividing the plane of the paper  
into four "quadrants," I, II, III, IV,  
we can associate  
every point in the plane with  
a pair of numbers, thus:



$(3,2)$  designates the point which is located 3 spaces to the right of O and two spaces up.

Similarly  $(8,4)$  is the point located 8 spaces to the right and 4 up.

Note that the first of the two numbers gives the distance along the X axis, and the second number of the pair gives the distance parallel to the Y axis.

And

if the first number is negative, like  $-2$ ,

we must go to the LEFT on the X axis instead of to the right.

And, similarly,

if the second number is negative we must go DOWN instead of up.

Thus

$(-4,-5)$  designates a point 4 spaces to the left of O and 5 spaces down

(see the diagram on page 96).

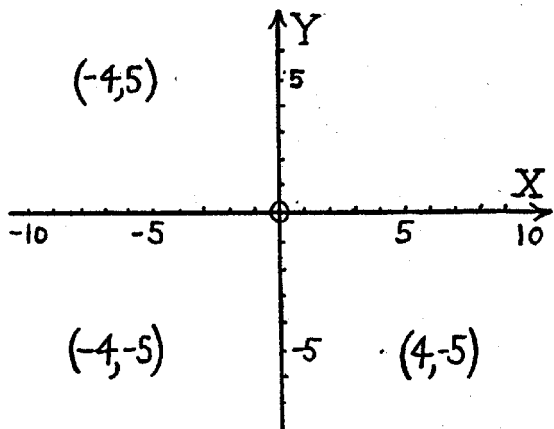
And of course

$(-4,5)$  means 4 to the left and 5 up,

$(4,-5)$  means 4 to the right and 5 down, and so on.

By this simple device

we can get a picture which

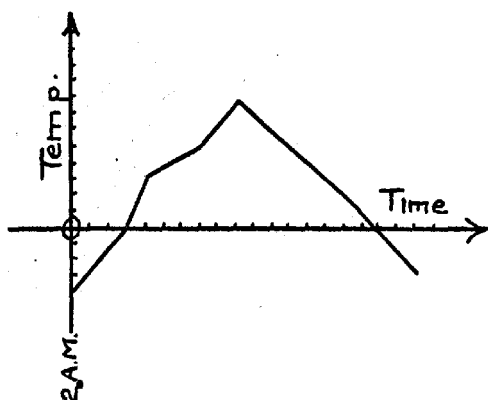


gives us the information that is often contained in columns of numbers, and gives us this information much more vividly.

For example,  
if the temperatures at a given place during a certain day are:

Time	Temperature
2 A.M.	$-4^{\circ}$
5 A.M.	$0^{\circ}$
6 A.M.	$3^{\circ}$
9 A.M.	$5^{\circ}$
11 A.M.	$8^{\circ}$
6 P.M.	$1^{\circ}$
9 P.M.	$-3^{\circ}$

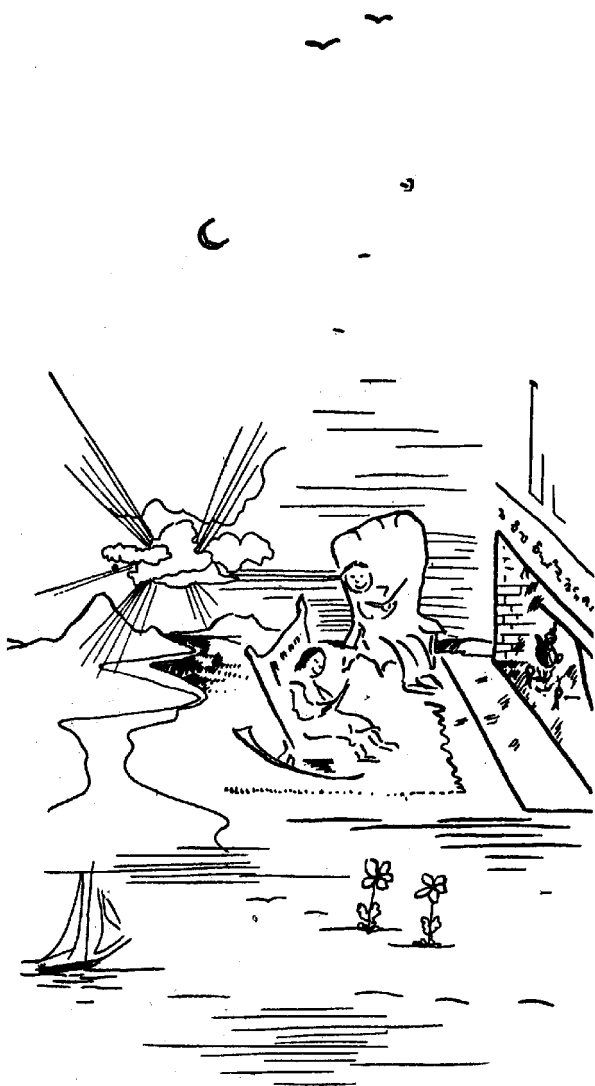
they may be represented  
“graphically” thus:



Graphs of this type are doubtless familiar to you, for the “practical businessman” has seen the tremendous advantage that this pictorial kind of representation has for his business, in advertising, in examining his volume of business, and so on and so on. A physician making his daily visit to a hospital can walk through a ward and see

each patient's temperature chart  
at a glance,  
and decide quickly where his  
special attention is required  
without wasting time to examine  
many columns of figures.  
All this is so familiar to T.C.  
that we need not emphasize further  
this debt that we all owe to  
the mathematicians for  
showing us this simple but  
practical device.

But while "practical" men  
have merely used this device  
for these simple purposes,  
the mathematicians have,  
by playing with the device itself,  
put it to infinitely greater use.  
We shall not give you here  
the details of how  
the mathematicians developed  
this simple device of a "graph"  
into a branch of Mathematics  
known as Analytic Geometry,  
without which  
we would not have had  
Newton's Calculus with its  
tremendously important applications  
to



Engineering,  
Physics,  
Chemistry,  
with the resulting benefits to us  
in  
Transportation via  
railroads and ships and planes;  
in  
Communication via  
telephone and telegraph and radio,  
and all the other benefits in  
diet,  
health,  
air-conditioning,  
etc., etc.  
For these are all described in  
other books,  
and there is no need to  
repeat these stories here.

The Moral: Why not read some or  
these stories in  
your spare time?

## X. THE OFFSPRING

We just want to indicate briefly here one major idea of Newton's Calculus:

Suppose you are taking a trip in an automobile and traveling at a steady rate of 40 miles an hour.

How far can you go in 2 hours?

Obviously the simple formula

$$(2) \quad d = rt$$

(distance = rate  $\times$  time)

will give you a quick answer.

But suppose that

your rate is not constant;

you can easily see that

this formula will no longer work.

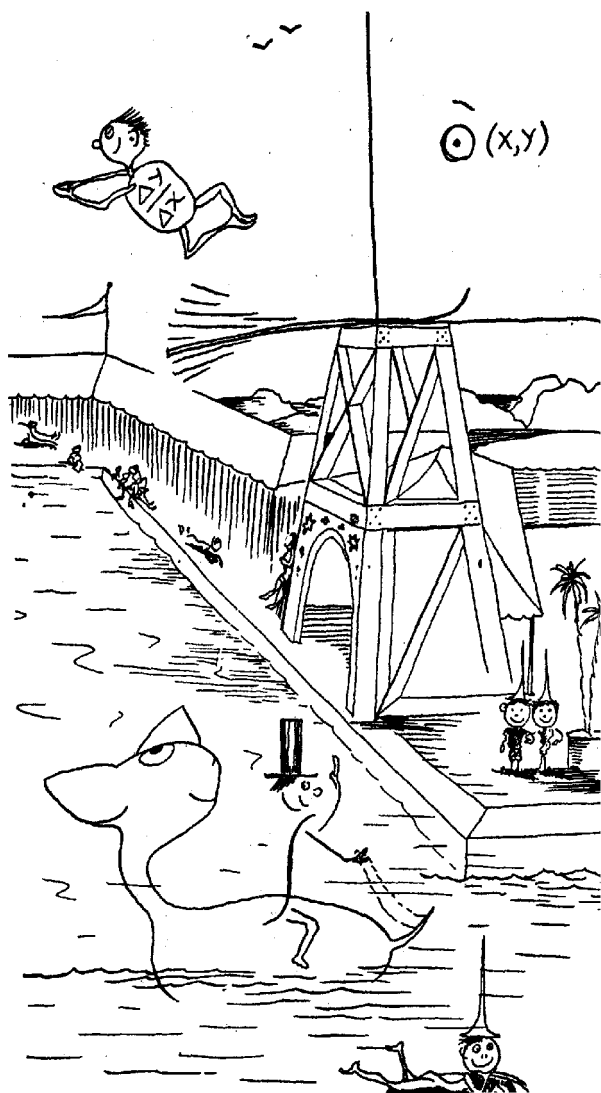
And since we often have need

for formulas which will apply

to motions in which

the rate is not constant,





let us see how this can be done.

To do this easily,

let us first

plot the graph of equation (2)

for the case when  $r = 40$ ,

namely

$$(3) \quad d = 40t.$$

We first make a table,

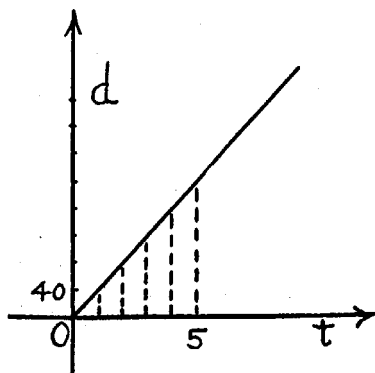
by giving  $t$  any values we please,

and calculating from (3)

the corresponding values of  $d$ :

$t$	$d$		
0	0	3	120
1	40	4	160
2	80	5	200

and then plot these points:



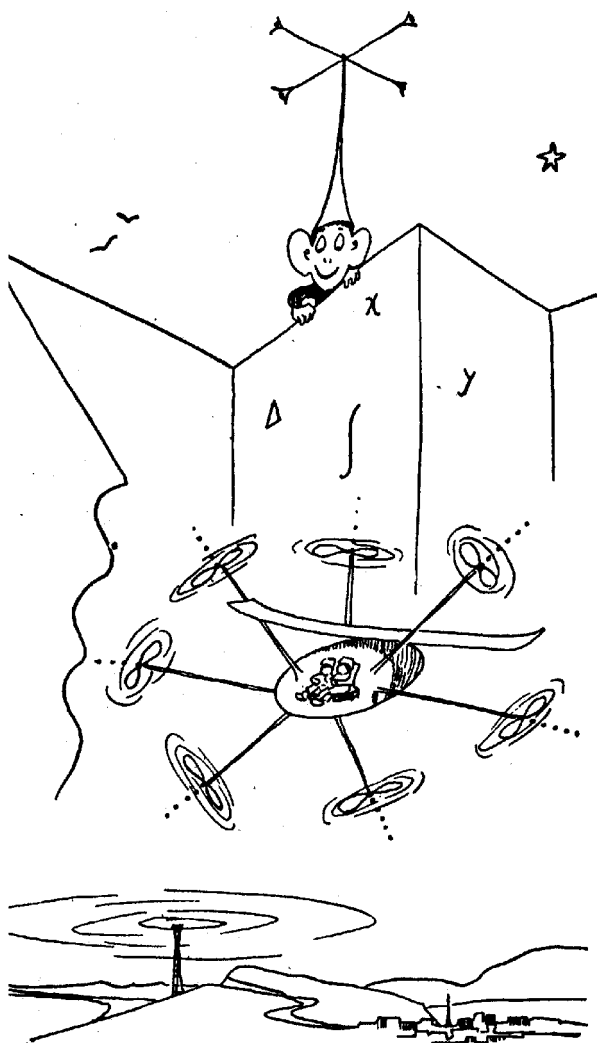
Now since equation (2)  
may be written

$$r = \frac{d}{t}$$

(to find the rate,  
divide distance by time),  
we see from the graph that  
the rate may be found by  
dividing the value of  
any dotted line  
(which represents distance traveled)  
by the corresponding value of  $t$ .

And so the graph on page 103  
completely shows  
the motion in question,  
the time being shown  
along the horizontal axis,  
the distance along  
the vertical axis,  
and the rate being their ratio.  
And obviously,  
a motion having a constant rate  
will be represented by  
a straight line.

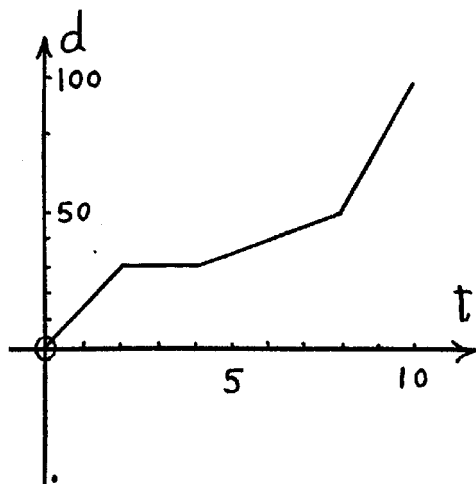
Now,  
what about a motion in which  
the rate is NOT constant?  
Suppose, for example, that  
you go at a rate of



20 miles per hour for  $1/2$  hour,  
 then increase your speed to  
 40 m.p.h.,  
 and keep that up for 2 hours,  
 then stop for an hour,  
 and then continue for 3 hours at  
 the rate of 35 m.p.h.,  
 what would the graph look like?  
 Obviously it would look like this:



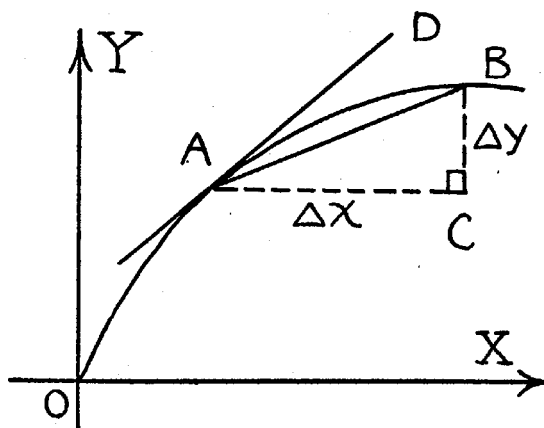
And, similarly,  
 the following "broken line" graph  
 tells what story?  
 For each straight portion of the line  
 the rate is uniform.  
 But at each CHANGE of slope of the line  
 the rate changes to  
 a new value which remains the same  
 until the next break.  
 Note that at each break  
 the change, as shown in these graphs,  
 is a sudden change,



no allowance being shown for the process of accelerating or slowing down.

To show this process,  
we must have a CURVE as shown  
on page 108,  
where  $x$  is the time and  
 $y$  the distance covered.  
Here any particular rate is  
not kept up for an appreciable time  
but is CHANGING ALL the time.

How can we now "catch" a thing



which is so elusive?

That was the problem solved by  
the Calculus:

Suppose first that  
the motion from A to B were  
a uniform motion instead of  
an accelerated one.

Then it would be represented by  
the straight line AB instead of  
the curve AB.

And it would show that  
in time AC  
the distance BC was covered,  
at a constant rate equal to  
 $BC/AC$ .

Now as you take the point B  
nearer and nearer to A,  
the straight line AB approaches  
more and more to the line AD  
which is tangent to the curve  
at point A.

Thus we may say that  
the actual rate at A is  
the "limit" of  $BC/AC$ .

And whereas  
this rate lasts only an instant,  
(for as soon as you get away from A  
the slope of the tangent line is  
obviously different),  
still

we can "catch" it and  
express it mathematically  
(and thus be able to work with it).

Thus,  
if we represent AC by  $\Delta x$   
(read "delta x"),  
which simply means the  
difference in the x-value  
from A to B,  
and BC by  $\Delta y$ ,  
then,  
as B approaches A,  
this ratio  $\Delta y/\Delta x$  approaches  
a limiting value.  
This limiting value of  $\Delta y/\Delta x$  is



represented by  $dy/dx$ .

And so we have

$$dy/dx = r,$$

the rate AT THE POINT A —

and  $r$  of course changes

from point to point.

Now,

if we know

the equation of the original curve,

the Calculus gives us

the necessary machinery

(called "Differentiation")

by which we may find

$$dy/dx$$

at any point.

And, vice versa,

if we know the value of  $dy/dx$ ,

that is, if we know the

"Differential Equation,"

we can find,

by means of the Calculus

(by "Integration"),

the equation of the original curve.

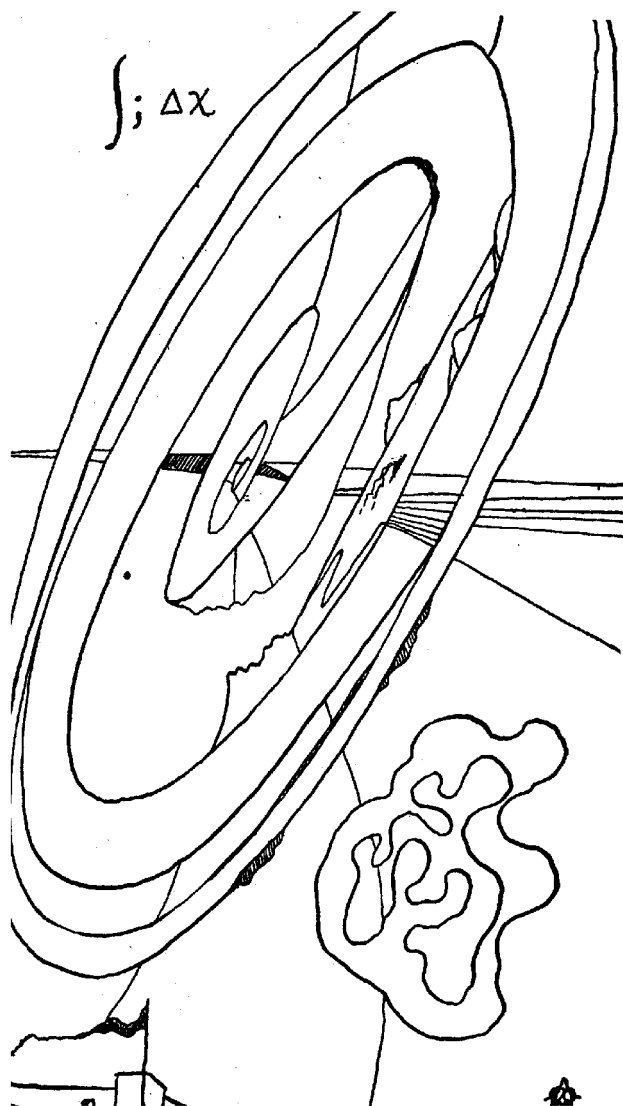
Now in most physical problems,

in this ever-changing world,

the idea is

to set up a differential equation

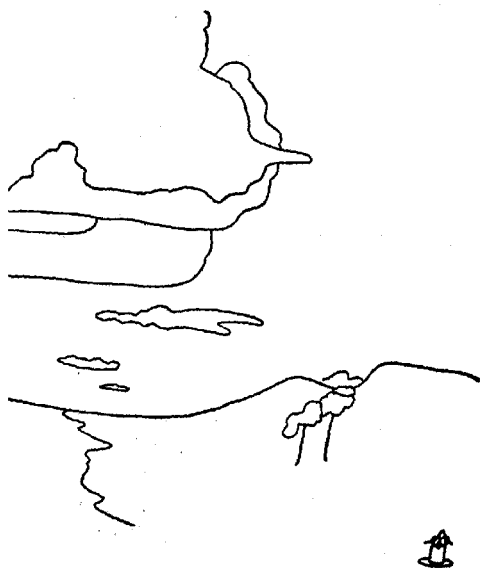
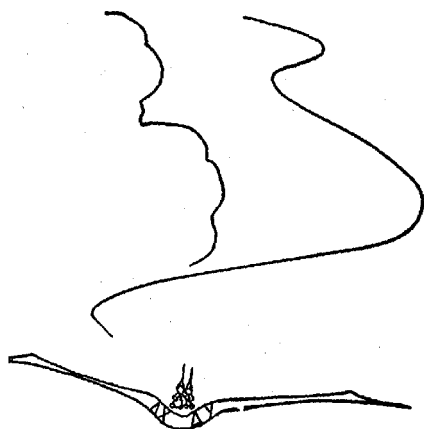
which represents what is happening



in a small local region,  
and from this,  
by "Integration,"  
to find out, for example,  
the entire sweep of  
the path of a planet.  
We can hardly expect,  
from this brief sketch,  
that anyone can get  
even a slight idea of the  
power of the Calculus  
as a tool of Science.  
Suffice it to say, here,  
that  
it is a method which  
enables us to study  
an ever-changing world,  
rather than only those things,  
like the figures in Geometry,  
which very accommodatingly  
stand still while  
we are measuring them.  
It is an instrument  
for the study of  
a swift, dynamic world.  
Why then is it not  
the last word in Mathematics?  
What more is there to be desired?

But wait till you see Part III!

The Moral: Learn to study  
ON THE WING!



## XI. A SUMMARY OF PART ONE

We have tried in Part I to  
give you the following ideas:

- (1) A man trying to think  
without Mathematics is  
like a helpless child  
(see Chapters I, II, and III) .
- (2) A "practical" man  
working with his hands alone,  
without the aid of theory,  
may be just a fool  
(see Chapters II, V, VI) .
- (3) The value of  
Mathematics and Science  
is not limited to  
the gadgets which they give us,  
but is also in their  
philosophy  
(see Chapters V and VI) .
- (4) Generalization and abstraction  
(two powerful tools of thought)  
are important in  
all thinking.

You cannot really think  
without them.  
But you must learn to use them  
properly.

If used carelessly  
they are "dynamite" and  
*may blow you up!*  
(see page 75 and page 67) .

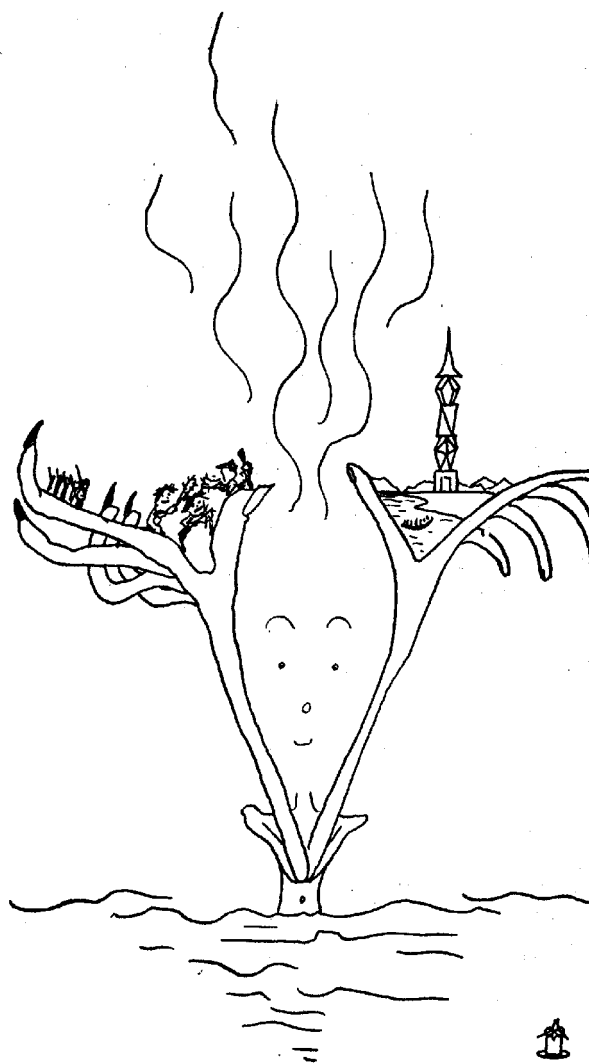
- (5) Do not always demand a  
"Yes" or "No" answer.

For example:  
"Shall we cling to the traditions  
of our great forefathers?  
Yes or no?"

The history of Mathematics shows  
just how much of Euclid  
we must keep  
and how much we must discard.  
You will see this in some detail  
in Part II.

But outside of Mathematics,  
in the social studies,  
you will hear people quoting  
blindly:  
quoting the Constitution,  
quoting Karl Marx,  
quoting Theodore Roosevelt, \*

By the way,  
the men who wrote the Constitution,  
as well as other men so often quoted,  
would be horrified at some of the  
applications made by their disciples.  
BEWARE OF DISCIPLES!



with the implication that  
you must either  
completely accept or  
completely reject.

In Mathematics, however,  
we do not just quote authority.

We say:

"In the light of our knowledge today,  
Euclid was right in this and  
wrong in that."

And this is a wholesome way  
to look at the past.

It is partly good and partly bad;  
we must select,  
in the best light of  
our knowledge now.

- (6) Do not jump at conclusions  
(see Chapters I, II, and III).
- (7) Do not rule out hunches  
because they are sometimes wrong.  
Read some of the original  
writings of Faraday or  
other great scientists—  
you will be surprised to find  
how much of their work  
started as a "hunch."

BUT

- (8) Do not think all  
your hunches are wonderful!  
Some of them may be terrible!



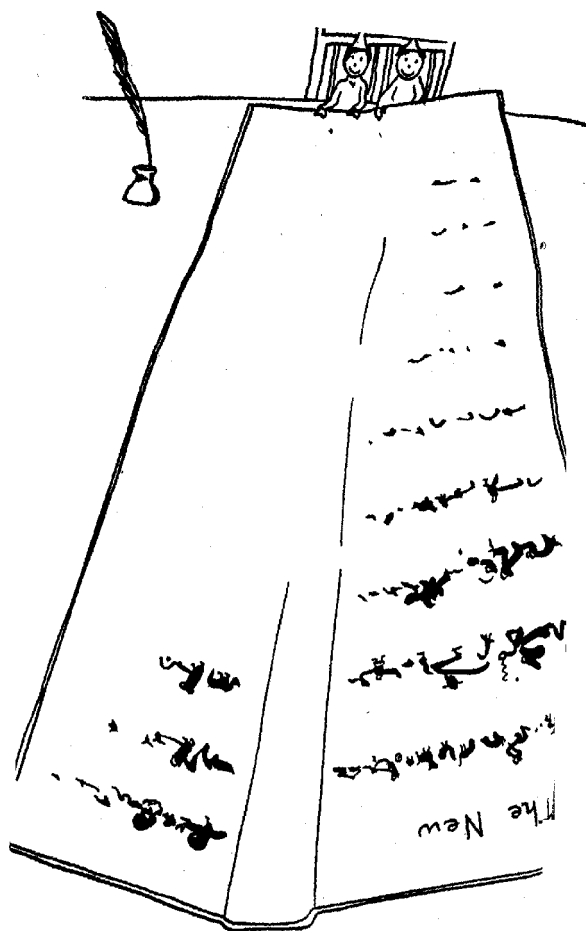
*Follow them up cautiously!*  
*Encourage them but*  
*watch them!*

- (9) Try to judge  
statements and theories  
in the light of  
important long-time activities  
of the human race—  
like Science or  
Mathematics or  
Art.  
They reveal “human nature”  
better than anything else.  
In them you will see that  
Internationalism and Democracy  
are very deep in the human spirit  
(see Chapter VI).

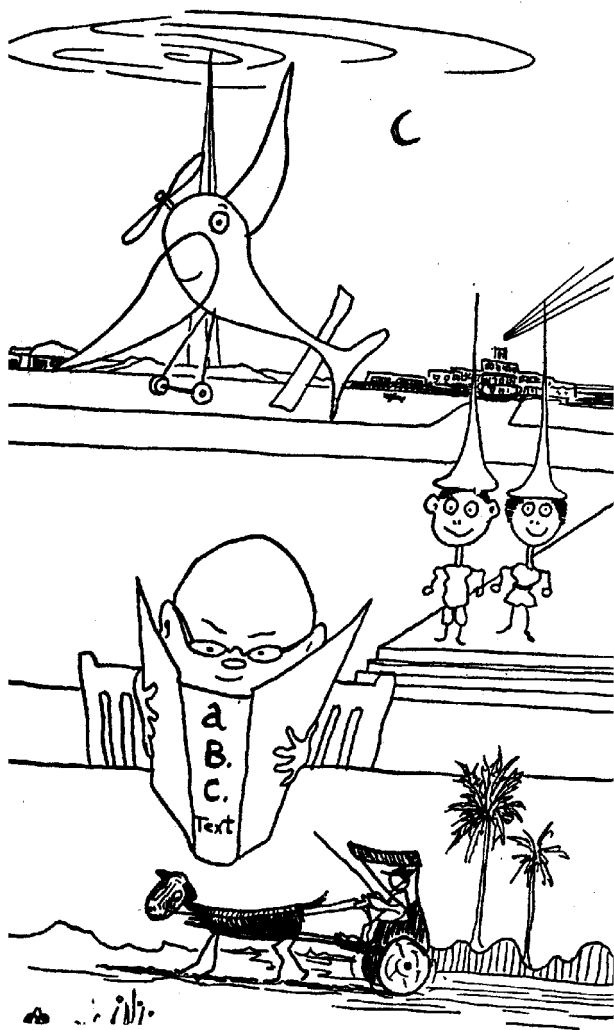
- (10) And so you see that  
Mathematics is not for  
the engineer only,  
or only for someone who  
needs its formulas.  
It is a way of thinking,  
a way of life,  
**VERY IMPORTANT FOR EVERYONE.**

- (11) Most courses in Mathematics  
do not leave time  
to consider all these things.  
They are too full of technique.  
We **MUST** stop now and then

from the manipulation of  
techniques  
to see what  
general ideas we can get from them,  
which will be useful for  
ALL of us.



**PART II**  
**THE NEW**



## XII. A NEW EDUCATION

And so you know that  
Algebra is a sort of  
Generalized Arithmetic  
by which more difficult problems  
may be solved.  
That Geometry is  
not only the study of  
various figures in  
two and three dimensions,  
but is also  
a sample science,  
the entire structure of which  
is built up from  
a few basic postulates—  
and is therefore a “model”  
for any system of thought.  
That Analytic Geometry is  
a combination of  
Algebra and Geometry which  
has proved extremely useful.

And that Calculus is  
a powerful instrument for  
the study of  
our DYNAMIC world.

You know also that  
Mathematics is useful not only  
as a technique,  
but also as a sample of  
a method of thinking:  
it is clear,  
precise,  
brief,  
many-sided.  
That a THOUGHTFUL study of  
even a little Mathematics  
can throw much light on  
many controversies,  
even with very little use of  
mathematical technique  
(see the summary of Part I).

Perhaps you may say:  
"What more can we ask?"

But the fact is that  
all the branches of Mathematics  
*mentioned in Part I*  
had been discovered  
by the time of Newton,

who lived from 1642 to 1727.  
And it was he who  
invented the Calculus.  
Analytic Geometry dates from Descartes,  
about 1637.  
Euclid goes back to  
about 300 B.C.  
And a good deal of the Algebra  
which is studied in  
high school and college  
is spread out  
from as far back as  
about 3000 B.C. to  
the time of Newton.

Thus,  
the knowledge of Mathematics  
of the average college graduate  
stops with what was known  
about 300 years ago!  
And yet  
more Mathematics has been invented  
in the last 100 years  
than in all the previous centuries  
taken together!  
If the same were true about  
the study of Physics,  
the average college graduate would  
never even have heard of  
an airplane or



an automobile or  
a radio,  
etc., etc.

Such a situation in Physics  
would never have been tolerated.

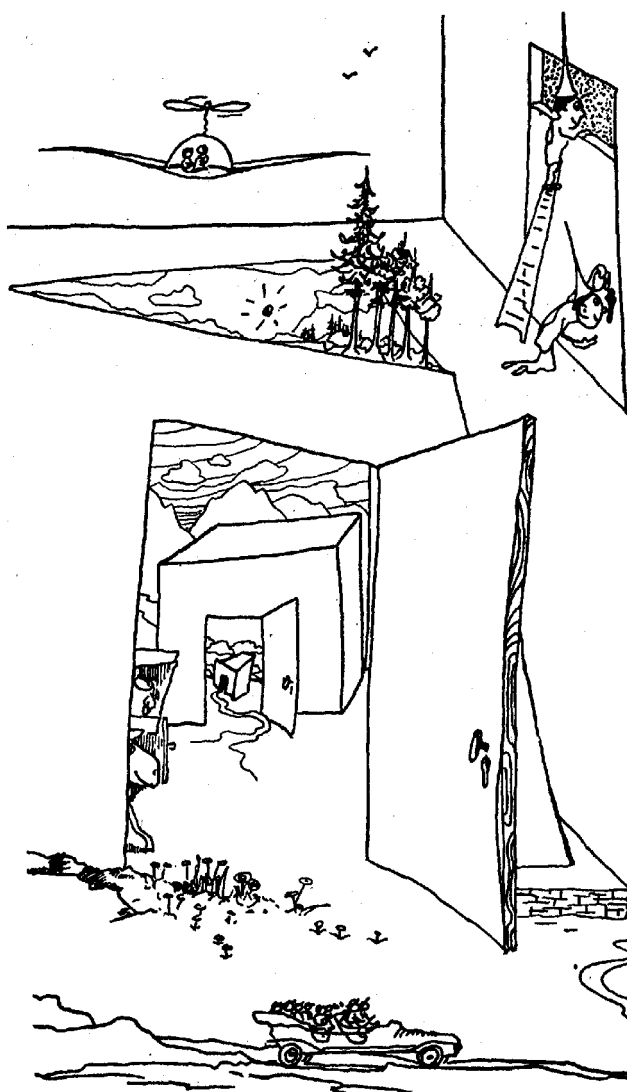
Why then is it tolerated in  
Mathematics?  
Perhaps MODERN Mathematics is  
so difficult that  
it can be understood only by  
a few rare souls?  
Not at all!  
Of course it took  
a few VERY RARE souls indeed  
to CREATE it,  
But these new results are  
no harder to understand than  
any of the older Mathematics.

Perhaps it is just inertia  
on the part of some educators?  
And T.C.,  
not being aware of  
what he is missing,  
does not clamor for it!

We feel that he can get even more of  
an intelligent, general outlook on  
life

from the MODERN ideas than  
from the older ones!

Read the next few chapters and  
see if you agree with us.



### XIII. COMMON SENSE

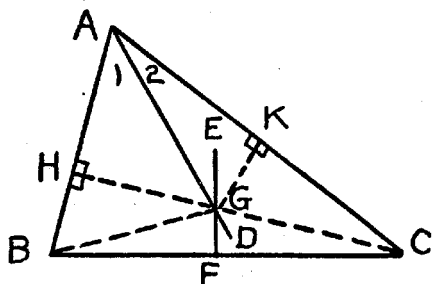
As we have already said,  
one of the chief values of  
the study of Geometry  
lies in the fact that  
it is a model for  
any science  
or for  
any system of thought,  
since it starts with  
a few basic ideas  
from which all the other ideas or  
"propositions"  
are derived by logic.

Now Euclid regarded these basic ideas  
as "self-evident truths";  
and  
some of them seemed to him  
so "self-evident" that  
he did not think it necessary

even to mention them.  
For example,  
he thought it was so obvious  
what is meant by  
the "inside" and the "outside" of  
a triangle  
that he did not bother to  
define them.  
And no doubt T.C. is thinking  
this minute  
that it is only "common sense,"  
and that any fool can SEE  
which is which  
at a glance.

But he will soon see the trouble  
this "common sense" caused,  
for we shall now show him that  
it led to the following absurd  
proposition:  
"If a triangle is NOT isosceles,  
then it MUST be isosceles"!  
(You remember of course that  
an isosceles triangle is one which  
has two of its sides equal.)  
In order to follow the proof  
you may have to recall  
some of your high-school Geometry—  
but that cannot hurt anyone, much.

Given, then, that  
 $AB$  does NOT equal  $AC$ ;  
 we shall now prove that  
 THEREFORE  
 $AB$  DOES equal  $AC$ !



First draw  $AD$  so that angle 1 = angle 2;  
 and let  $FE$  be  
 the perpendicular bisector of  $BC$ .  
 Now, if the triangle were isosceles,  
 $AD$  and  $EF$  would be  
 one and the same line,  
 but since the triangle is NOT  
 isosceles,  
 $AD$  and  $EF$  must intersect.  
 Let us call their point of intersection  $G$ .  
 And now draw  $BG$  and  $CG$ ;  
 also draw  $GH$  perpendicular to  $AB$ ,  
 and  $GK$  perpendicular to  $AC$ .  
 Now  $BG = CG$  because  
 any point in the

perpendicular bisector of a line  
is equally distant from  
the ends of the line.

(This is a well-known proposition  
in Geometry,  
remember?)

Also  $GH = GK$  because  
any point in the  
bisector of an angle is  
equally distant from  
the sides of the angle.

(Another well-known proposition—  
perhaps you had better have  
your Geometry book handy!)

This makes  
triangle  $BGH$  congruent to  
triangle  $CGK$ ,  
since

two right triangles are congruent if  
the hypotenuse and a leg of one  
are equal respectively to  
the hypotenuse and a leg of the other.

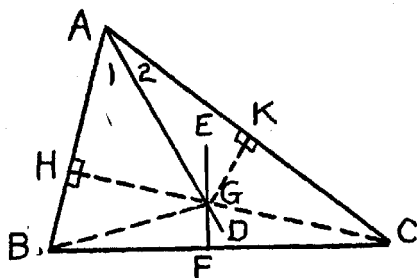
(Another call for that Geometry book!)

Therefore  $BH = CK$  (1)

for  
corresponding parts of  
congruent figures  
are equal.

Similarly  
triangle  $AGH$  is congruent to  
triangle  $AGK$ ,

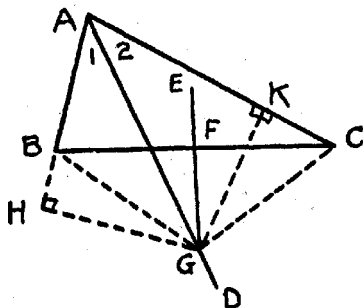
and consequently,  $AH = AK$ . (2)  
Hence, adding (1) and (2) above,  
we get  $AB = AC$ ,  
showing that  
two UNEQUAL sides of a triangle  
MUST be EQUAL!



You probably do not like this result  
any more than  
the mathematicians did.  
Now,  
if you remember your Geometry  
pretty well,  
you will immediately say that  
you know just where the trouble is:  
namely,  
that  $AD$  and  $EF$  intersect all right  
BUT  
NOT as shown in the diagram—  
that they really intersect



outside the triangle,  
like this:



Here again  
triangle BGH is congruent to  
triangle CGK,  
making  $BH = CK$ .

And  
triangle AGH is congruent to  
triangle AGK  
so that  $AH = AK$ .

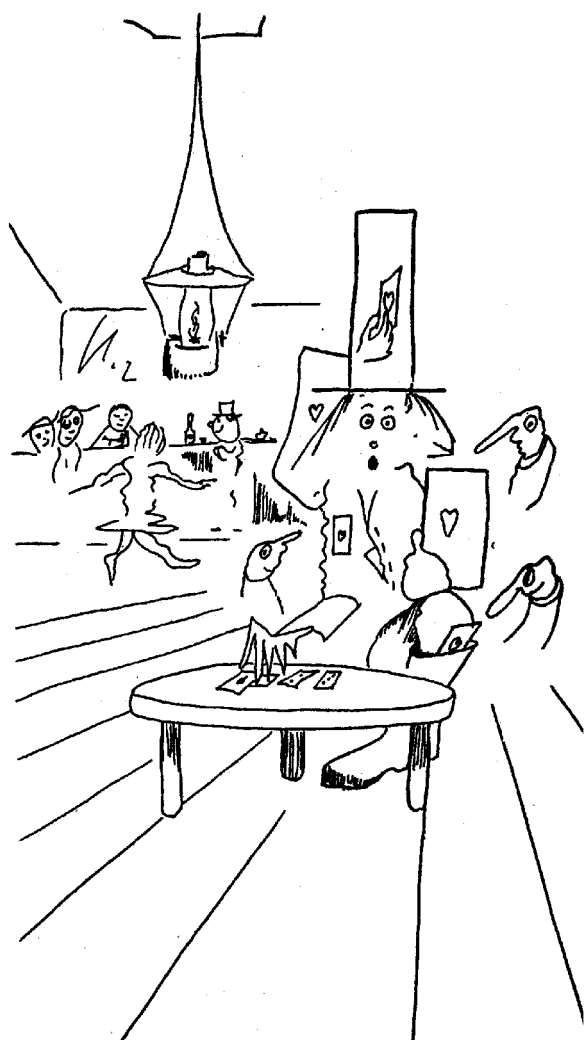
But now this does NOT make  $AB = AC$ ,  
since  $AH + BH$  NOW does NOT equal  $AB$ ,  
as it did before  
(although  $AK + KC$  still equals  $AC$   
as before).

Hence now  
we do NOT get the absurd conclusion  
we got before.

BUT  
we cannot let you off so easily!

Because  
you are using the DIAGRAM  
to prove your point,  
instead of LOGIC!  
Perhaps you will demand to know  
"What is the difference?!"

Well,  
the difference is that  
diagrams are NEVER used as  
evidence in Geometry.  
Why?  
Because Geometry is not  
that kind of subject.  
It is a subject in which  
the theorems are derived from  
the basic postulates  
by means of LOGIC.  
And if there is no definition given  
of "outside" and "inside" of  
a triangle,  
no argument can be based on  
such a nonexistent definition.  
Do you think this is just  
quibbling?  
But mathematicians have been fooled  
by this sort of thing before,  
and are more cautious now—  
and that is what we recommend  
to T.C. also.  
For how would he like to



be accused of a crime which  
is not even recorded in  
a law book?  
Would he not then appreciate  
a lawyer who would argue  
that there is no validity in  
a "tacit" law?

Thus mathematicians too  
have learned by hard experience  
not to base an argument on  
"tacit" assumptions.

Perhaps this brings to your mind  
cases from modern psychology,  
in which much damage is done  
to an individual's nervous system  
by "subconscious" ideas,  
which, if brought up  
into consciousness,  
can be treated and  
eliminated as a source of difficulty.

Thus, one modern trend seems to be  
to turn the light on  
our subconscious thoughts and  
rid ourselves of the  
prejudices and false thinking  
which may be due to them.

The Moral: Be REASONABLE  
by bringing to light  
your "tacit" ideas.

#### XIV. FREEDOM AND LICENSE

As we just saw,  
one of the tasks of  
modern mathematical research  
has been  
to go back to Euclid,  
bring to light his  
"tacit" assumptions,  
and make the kind of "phoney proof"  
shown in the last chapter  
IMPOSSIBLE,  
at least in Euclidean Geometry.  
And is it not up to us  
to take a leaf from  
the mathematician's book,  
and profit by his experience,  
by trying to make  
"phoney proof" impossible also in  
other domains of argumentation,  
by not permitting any  
"tacit" assumptions?

But this is not all!

What about the  
"self-evident truths"  
which were not tacit,  
but were expressly stated?

Well,  
one of these  
even at that time  
did not seem so "self-evident":  
namely, the one which says that  
"Through a given point,  
which is not on a given line,  
one and only one line can be drawn  
parallel to the given line" —  
known as the famous  
"parallel postulate."  
Not considering it to be  
"self-evident,"  
Euclid tried to prove it,  
but without success,  
and finally listed it as  
"self-evident,"  
although he did not feel  
so good about it.

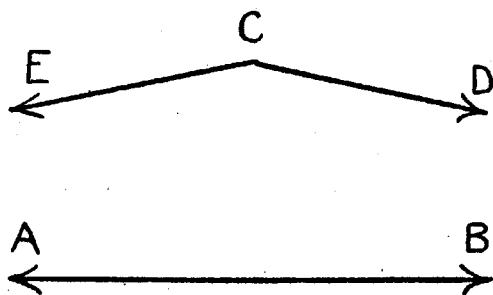
Then,  
for several hundred years,  
outstanding mathematicians again



tried to prove it,  
but again without success.  
FINALLY,  
but NOT UNTIL 1826,  
a remarkable thing happened.  
The idea dawned in the minds of  
several mathematicians at the  
same time  
(Lobachevsky, Bolyai, Gauss)  
that  
not only is this statement  
NOT "SELF-EVIDENT,"  
but in what sense is it  
"true" at all?  
And so they undertook  
to see what would happen if  
they assumed that  
"Through a given point  
(not on a given line),  
TWO lines could be drawn  
parallel to the given line,  
one to the right and  
a different one to the left."

Perhaps T.C. will immediately object  
and say,  
"But this is impossible";  
and will get all excited and  
draw a diagram like this:





and say:

“Can’t you see that  
if you draw  
two distinct lines through C,  
as shown,  
neither one of them can be  
parallel to AB?  
For CD will meet AB somewhere  
to the right,  
and CE will meet it on the left.  
A little common sense  
shows this so clearly.”

But we must warn T.C. against  
that “common sense” of his;  
not that it isn’t a good thing,  
but he must use it with  
**SO MUCH MORE CAUTION**  
than he realizes

(don't forget Chapters I, II, III  
in Part I,  
and page 93).

And we must also warn him again  
against this reckless use of diagrams,  
and compel him to  
stick to LOGIC  
as his SAFEST weapon for  
clear thinking.

Now when these three  
very intelligent men,  
mentioned above,  
looked into the matter  
(quite independently of each other,  
by the way),  
they found that  
no logical fallacy resulted from  
their strange assumption,  
but that they got  
an entirely DIFFERENT Geometry!  
A queer-sounding Geometry in which  
the sum of the three angles of  
a triangle  
was no longer  $180^\circ$ ,  
in which  
the famous Pythagorean Theorem  
was no longer true,  
and yet  
the LOGIC was PERFECT!

*Well, you may say,*  
*"So what?*  
*A couple of wild-eyed,*  
impractical mathematicians  
go haywire,  
lose all common sense,  
fool around with their  
silly logic,  
get ridiculous results,  
and I, T. C.,  
should get excited about it?"

Well, T.C.,  
what would you say  
if we should tell you that  
in 1868  
a man named Beltrami found that  
all this stuff was not just  
fantastic nonsense  
but actually applied on a surface  
called a "pseudo-sphere"?!

And he then understood that  
whereas  
good old Euclidean Geometry  
applies on a flat surface,  
like an ordinary blackboard or  
a piece of paper,  
that other Geometries are needed  
for other surfaces,  
and that it is all  
very sensible.

Just as, for example,  
the Geometry on the  
surface of the earth  
is also non-Euclidean,  
since here too  
the sum of the angles of a triangle  
does NOT equal  $180^\circ$ :  
thus consider the triangle  
formed by an arc of the equator and  
the parts of two meridians drawn  
from the north pole to  
the ends of this arc;  
the two base angles here are  
each equal to  $90^\circ$ ,  
so that all three angles together  
do NOT add up to  $180^\circ$ .

What, then, about  
those "self-evident truths"?  
Apparently BOTH  
Euclid's parallel postulate AND  
the non-Euclidean parallel postulate  
mentioned above  
(permitting TWO parallels through  
a given point)  
are equally true,

\* See "Non-Euclidean Geometry"  
by Hugh Gray and Lillian R. Lieber  
(Galois Institute Press).

one of them applying on one surface,  
the other on a different surface!

And so gradually  
the mathematicians have been forced  
to the position that  
in Mathematics,  
instead of regarding  
the fundamental postulates as  
"self-evident truths"—  
as Euclid did—  
it makes more sense,  
in the light of the experience  
described above,  
to regard them as  
mere ASSUMPTIONS.

In other words,  
the mathematician now would say  
if I assume some things  
and then derive from them  
other things by the use of  
Logic,  
WITHOUT CONTRADICTING myself,  
that is all I ask.  
For, fundamentally,  
I am not even concerned with  
the question of actually  
finding a surface for which  
a certain Geometry holds;

for my job is to find out  
what straight thinking can do.  
In a way it is really a  
psychological problem.  
And I find that  
in some ways  
I have a good deal of freedom;  
and, in other ways,  
I am bound tight:  
thus, I am  
**FREE TO SELECT ANY  
BASIC ASSUMPTIONS I PLEASE  
EXCEPT  
that they  
MUST NOT CONTRADICT EACH OTHER.**  
In this way,  
I can develop  
all kinds of  
systems of thought..  
I find that  
a great many of them,  
often to my own surprise,  
actually find applications  
in the physical world.  
And, from past history,  
I feel that  
many more of them will find  
applications in the future.  
But I am really driven by  
a great curiosity and delight in

*the truly remarkable worlds*  
I create.  
They may sound fantastic to  
the uninitiated,  
but I find them not only  
fascinating but  
they also throw so much light on  
"Just what is human thinking anyway?"

Note that he has a very clear  
realization of  
where freedom ends and  
license begins: \*  
he knows full well that  
he cannot introduce anything  
into a system which would  
destroy the system itself  
by contradiction.

Page those pseudo-liberals  
who try to introduce into  
a system of Democracy  
ideas which would  
destroy Democracy itself.  
They ought to be  
**OBLIGED TO STATE EXPLICITLY**

\* See that delightful book by  
C. J. Keyser  
called: "Mathematical Philosophy,  
A Study of Fate and Freedom" (Dutton).

JUST WHAT THEY CONSIDER TO BE  
THE FUNDAMENTAL POSTULATES FOR  
DEMOCRACY;

they would probably find that  
a good many things they now advocate  
CONTRADICT

some of their fundamental ideas.

And they would be obliged to admit  
that

even FREEDOM OF SPEECH itself  
is necessarily limited,

since it must not be used  
to contradict

the other postulates for Democracy.

So that even in a Democracy

it is NOT LOGICAL to

allow an enemy of Democracy to use

Freedom of Speech

to destroy Democracy!

Similarly

FREEDOM OF ENTERPRISE

must also necessarily be limited

by the other postulates of Democracy.

And so you see how

Mathematics can throw light

on various subjects

which many people discuss

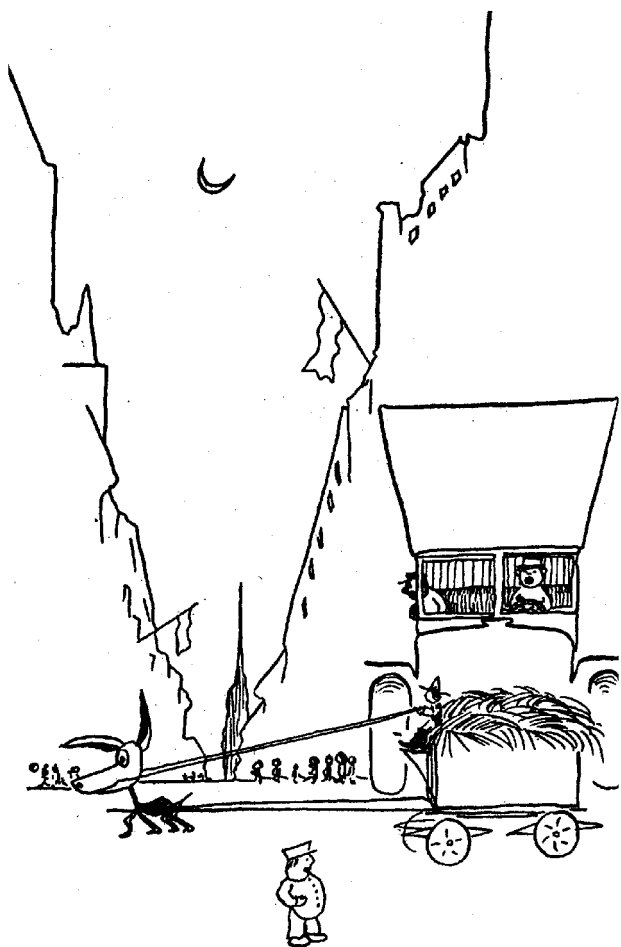
glibly and carelessly





since they have never been trained  
to examine ideas  
With that METICULOUS CARE  
with which a mathematician works.  
THERE is a model for straight thinking  
which we all MUST try to imitate.  
This is not the  
noisy argumentation of  
the pseudo-thinkers.  
Rather it is  
quiet,  
honest,  
careful,  
COMPETENT.

The Moral: Do not be NAÏVE—  
use the methods of  
Mathematics.



## XV. PRIDE AND PREJUDICE

We hope that by this time  
you no longer feel that  
"There is only one Geometry—  
good old Euclid;  
he may, have given me  
many a headache  
in high school,  
but at least he is respectable."  
And that you will not be inclined  
to agree with Mrs. Hardy in  
"Andy Hardy Meets Debutante"  
when she says:  
"Nice people never change";  
but that you are prepared to agree  
with the late  
Supreme Court Justice  
Benjamin N. Cardozo  
who said:  
"We are to beware of the  
insularity of mind that perceives

in every inroad upon habit  
a catastrophic revolution."

But, if you are still in doubt,  
we want to give you,  
in this chapter,  
a simple but most charming  
illustration of a Geometry which  
will limber up your mind  
so beautifully that  
you will be prepared to  
glide through this changing world  
with ease.

We must however call to  
your attention  
the fact that  
whereas  
new ideas are many and welcome  
in Mathematics,  
still  
they are not just the  
ravings of any "radical" child.

With this brief reminder,  
let's go.

You know, of course, that  
in Euclidean Geometry,  
a plane, or even a line,

has an infinite number of points.  
 Now, in the Geometry which  
 we shall presently describe,  
 this is not so;  
 here  
 there are only  
 TWENTY-FIVE POINTS  
 in the entire Geometry;  
 and it is therefore called  
 A FINITE GEOMETRY.

Let us designate the  
 25 points by the  
 25 letters of  
 the English alphabet,  
 from A to Y inclusive.  
 And let us arrange these letters  
 in three blocks as follows:

A B C D E	A I L T W	A X Q O H
F G H I J	S V E H K	R K I B Y
K L M N O	G O R U D	J C U S L
P Q R S T	Y C F N Q	V T M F D
U V W X Y	M P X B J	N G E W P

Now let us make the  
 following  
 ASSUMPTIONS:

- (1) A "straight line" shall mean  
 any row or column in  
 any of the three blocks above.

- (2) A point-pair shall be considered "congruent" to another point-pair when both pairs occur in rows (or both in columns), AND IF the number of steps between the points is the same in both pairs.

A	B	C	D	E	A	I	L	T	W	A	X	Q	O	H
F	G	H	I	J	S	V	E	H	K	R	K	I	B	Y
K	L	M	N	O	G	O	R	U	D	J	C	U	S	L
P	Q	R	S	T	Y	C	F	N	Q	V	T	M	F	D
U	V	W	X	Y	M	P	X	B	J	N	G	E	W	P

Thus,

$AB$  is congruent to  $HI$ ;

$QS$  is congruent to  $MX$

(even though  $QS$  is in a row of the FIRST block,

whereas  $MX$  is in a row in the SECOND block);

$AK$  is congruent to  $WD$ ;

etc.

But

$AB$  is NOT congruent to  $GI$ ,

etc.

Note also that

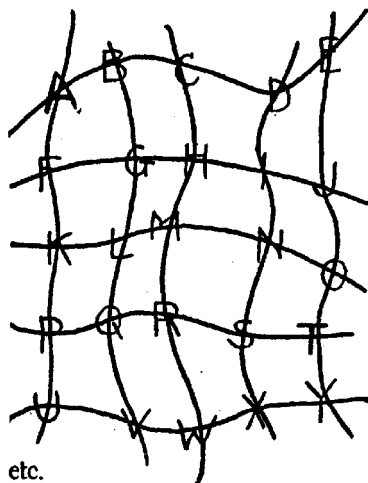
$AB$  is congruent to  $TP$   
because we shall consider  
the number of steps from  $T$  to  $P$   
(in the first block)  
to be one:  
for, when we come to  
the end of a row (or column)  
we continue to count forward  
by jumping to the beginning of  
the same row (or column).

Note also that  
 $AB$  is NOT congruent to  $AF$   
since  $AB$  is in a row,  
and  $AF$  is in a column,  
whereas in (2) on page 156  
it was said that  
both must be in rows,  
or both in columns,  
but not  
one point-pair in a row and  
the other in a column.  
Note also that the word  
"congruence" here  
does NOT mean the same thing as  
in Euclidean Geometry,  
where "congruence" involves  
"distance,"  
and two line-segments are  
congruent only if they



whereas here,  
there is no question of  
“distance” or “fitting,”  
but merely of  
“number of steps.”

Also,  
“straight line” here does not have  
the same significance as in  
Euclidean Geometry—  
since here it means only  
any row or any column.  
It would be better,  
in order to emphasize  
these distinctions,  
to arrange the blocks  
as follows:



And, surely, now  
you are not in the least  
disturbed by  
having *ABCDE* called  
a "straight line."

Why?

Because this is according to  
the "rules of our game":  
see (1) on page 155.

Let us now say that  
two straight lines are  
"parallel" if they have  
no point in common.  
This is a pretty good use of  
the word "parallel,"  
is it not?

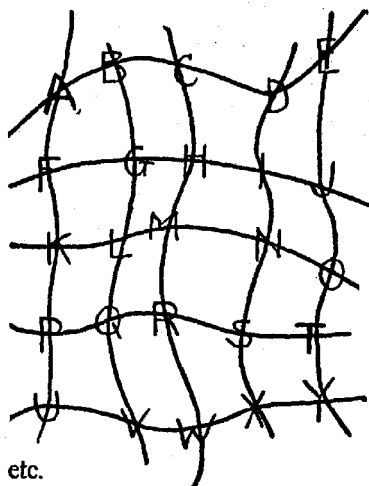
Hence,

*KLMNO* is parallel to *FGHIJ*,  
since none of the letters in *KLMNO*  
occurs also in *FGHIJ*;  
but

*ABCDE* is NOT parallel to *BGLQV*  
because they both have  
the point *B* in common.

Here of course  
there is no question of  
two lines meeting if  
"sufficiently prolonged,"

because here  
 no prolonging is possible,  
 since there are no points beyond  
 those we have exhibited;  
 the entire set of points  
 is visible at a glance.



Note that  
 whereas in Euclidean Geometry  
 two lines which are "parallel"  
 not only have no point in common,  
 but also

they are  
"everywhere equally distant,"  
but here,  
where "distance" has no significance,  
this second property disappears,  
so that two such lines as  
*BGLQV* and *EJOTY*  
may be considered parallel  
without worrying us at all.

In other words,  
one way in which  
the mathematician is enabled  
to make up a new system  
is  
to take some  
old familiar word,  
like "parallel,"  
examine into its various properties,  
retain some of these but  
discard others,  
thus obtaining  
a new freedom  
without entirely  
cutting loose from the past.

There may be a moral here for T.C.—  
a hint of  
how to proceed when  
looking for something new:

Not to break entirely with the past,  
but to mold it and modify it  
to suit new needs.  
Remember that  
an entirely new Geometry was found  
by merely changing  
ONE POSTULATE (see page 143)!

Let us now see  
what a triangle looks like  
in this new setup.  
Take for instance  
the triplet of points,  $H$ ,  $L$ , and  $R$ ;  
they form a triangle  
whose vertices are of course  $H$ ,  $L$ ,  $R$ ;  
and whose sides are  
 $HL$ ,  $LR$ , and  $HR$ .  
And since straight-line-segments  
are taken only along  
rows or columns,  
but not diagonally,  
we find the side  $HL$   
in the THIRD block on page 156,  
 $LR$  in the SECOND block,  
and  $HR$  in the FIRST block.  
Thus the triangle here  
is completely dismembered,  
like the lady in  
Picasso's "L'Arlésienne."  
Incidentally

it so happens that  
*HL*, *LR* and *HR* are all congruent  
 (since they are all in columns  
 and each is a two-step pair).  
 So that our triangle is equilateral;  
 whereas triangle *ABJ* is isosceles  
 but not equilateral,  
 and triangle *AST* is neither.

A circle here is defined, as usual,  
 as a set of points such that  
 any one of them taken with the center  
 gives a point-pair which is  
 congruent to every other such pair:  
 thus,  
 if we take *A* as center,  
 and *AB* as radius,  
 then points *B*, *E*, *I*, *W*, *X*, and *H*  
 all lie on the circle,  
 because *AB*, *AE*, *AI*, *AW*, *AX*, and *AH*  
 are all congruent;  
 so that here  
 a circle has only six points  
 on its circumference.

A	B	C	D	E	A	I	L	T	W	A	X	Q	O	H
F	G	H	I	J	S	V	E	H	K	R	K	I	B	Y
K	L	M	N	O	G	O	R	U	D	J	C	U	S	L
P	Q	R	S	T	Y	C	F	N	Q	V	T	M	F	D
U	V	W	X	Y	M	P	X	B	J	N	G	E	W	P

*Now you would be surprised to find  
that nearly all  
the Euclidean postulates  
are meaningful here,  
in spite of the  
meagerness of this little Geometry;  
and a great many of  
the Euclidean theorems  
hold here also.*  
For example,  
the three altitudes of a triangle  
are concurrent;  
so are the three  
perpendicular bisectors of the sides;  
and also the three medians.  
Furthermore  
the point of concurrence of the medians  
lies on the line joining  
the other two concurrency points  
mentioned above,  
and divides it in the ratio 2 : 1  
just as in Euclidean Geometry.

Here also:

If two sides of a quadrilateral  
are congruent and parallel,  
so are the other two sides.  
The diagonals of a parallelogram  
bisect each other.  
The diagonals of a rhombus

are perpendicular.  
At each point of a circle,  
there is one and only one "tangent"  
(that is,  
only one line which has  
a single point in common with  
the circle) .

In fact  
a whole theory of conic sections  
is possible here,  
etc., etc.

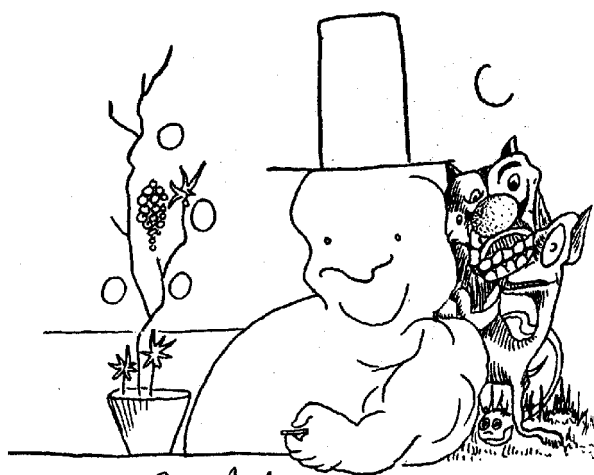
Do we hear some cynic say again:  
"So what?"

If so,

let us point out:

- (1) This Finite Geometry actually  
arises in connection with  
certain problems in  
Algebra and Number Theory!  
(You see, Mr. Cynic,  
do not be in a hurry  
to call a thing useless—  
perhaps you think it is useless  
only because your knowledge  
is limited!)
- (2) Note that  
this entire subject is built up  
by logic alone,





without diagrams—  
thus, a triangle,  
as we said before (see page 162)  
no longer looks like a triangle;  
circles look very STRANGE  
(completely dismembered!),  
and so on.  
And all this HELPS to  
EMPHASIZE relationships  
without OLD PREJUDICES!  
And to realize that  
Geometry is really a matter of  
LOGIC and NOT of DIAGRAMS!

The Moral: Beware of  
superficial appearances!  
Get behind them  
with a clear head,  
and find out  
what is back of that  
good old propaganda.  
This process may lead you  
to some strange,  
“dismembered,”  
modernistic things;  
but do not let the  
strangeness scare you;  
DEEP-SET PREJUDICE  
MAY BE WORSE THAN  
STRANGENESS!

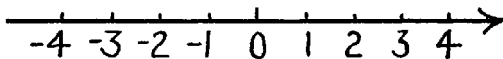
## XVI. TWICE TWO IS NOT FOUR!

Perhaps you have now become  
so modernistic that  
you are really no longer disturbed  
by the funny, dismembered  
triangles and circles,  
and that you are even  
willing to grant that  
there is some advantage in all this.  
But  
"Twice two is not four"!!  
This is really too much.

Let us try to make this clear.  
As we have seen,  
one way of making progress in  
Geometry  
is to take some old familiar word,  
like "parallel,"  
and limber up its meaning  
just enough to make it possible to  
put it to some new use  
(see page 161).

Now, perhaps you do not realize it  
but  
you have already had  
similar experiences in Algebra:  
For instance,  
when you began the study of Algebra  
in your first year in high school,  
you became acquainted with  
“negative” numbers.  
as distinguished from  
the ordinary, “positive” numbers of  
Arithmetic.

Thus, if we represent numbers by  
points on a line,  
we may place the positive numbers  
to the right of zero, thus:



and the new negative numbers  
to the left.  
So that in Algebra  
you were introduced to  
a whole new set of numbers which  
do not come into Arithmetic at all;  
and, as you doubtless know,  
these new numbers are just as

“practical”  
as the old ones,  
since, for example,  
a temperature of  $5^{\circ}$  below zero ( $-5^{\circ}$ )  
is just as “real” as  
a temperature of  $5^{\circ}$  above zero—  
ask Admiral Byrd!  
And a debt of 50 dollars ( $-\$50$ )  
is just as “real,” although not as pleasant, as  
having 50 dollars in your pocket—  
ask any man who is being sued for  
debt!

And you remember you then had to  
find out  
what is to be meant by  
“adding” these new numbers,  
and “multiplying” them.

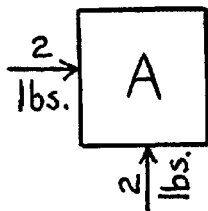
Perhaps you remember these  
new definitions,  
and perhaps you remember also  
how strange they seemed to you  
at the time.  
Thus the rule for “adding”  
a positive number and  
a negative number is:  
“Take the DIFFERENCE of the numbers  
and then prefix to the result  
the sign of the larger one.”

Or, in plain English,  
if you wish to "add"  
—5 dollars and 3 dollars,  
the answer is —2 dollars,  
because  
if you OWE 5 dollars ( $-\$5$ )  
and HAVE 3 dollars ( $\$3$ )  
and wish to  
BALANCE YOUR ACCOUNT,  
you pay part of your debt and  
still OWE 2 dollars.  
In other words,  
"addition" now means  
"balancing accounts."  
And you are not in the least  
disturbed by the fact that  
when you "add," you sometimes  
really "subtract,"  
as you may have heard some  
youngsters say in high school!  
The fact is that they are using  
"add" in the algebraic sense,  
and "subtract" in the  
arithmetic sense;  
but there is really no confusion  
if you realize clearly that  
"add" in Algebra does not have  
the same meaning as  
"add" in Arithmetic.

*Thus,*  
as we said before,  
you have already had some experience  
in changing the meaning of  
a word in Mathematics  
in order to have  
progress.

And those of you who have had  
some elementary Physics  
are familiar with  
still another meaning of the word  
“add”:

For instance,  
suppose a force of 2 pounds  
acts on a body, A,  
in the direction indicated;  
and suppose another force of 2 pounds  
acts on A in another direction,  
as shown:

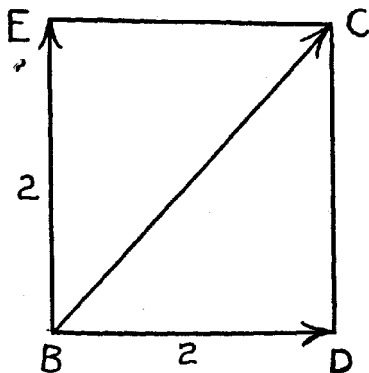


The question is  
in which direction will A

actually move,  
and how great a force is  
actually pushing it?

You probably remember that  
this problem is solved by  
what is known as  
"The parallelogram of forces,"  
as follows:

Draw two lines,  $BD$  and  $BE$ ,  
representing the two given forces  
in magnitude and direction,  
thus:



Then complete the parallelogram by  
drawing  $DC$  and  $EC$ ;  
then the line  $BC$  is



*the "resultant" or "SUM" of*  
the two given forces;  
and,  
from triangle *BDC* it is quite easy  
to calculate the length of *BC*,  
which in this case is about  
2.83 lbs.

So that here  
 $2 + 2$  is NOT 4 but 2.83!  
And,  
if the angle at *B* did not  
happen to be a right angle,  
the sum of 2 and 2 would be  
something else.

Thus,  
in "adding" forces in Physics,  
we must take this angle  
into consideration,  
and for every different angle  
we get a different answer for  $2 + 2$ !  
And yet you will admit that  
there is no confusion here  
whatsoever.

It makes perfectly good sense,  
does it not?

Now,  
realizing more than ever  
that in Mathematics we are free  
to make any assumptions which  
we find useful,  
so long as they

## DO NOT CONTRADICT EACH OTHER,

we see that in this way  
many different Algebras,  
as well as different Geometries,  
can be constructed—  
and actually have been.  
Some of these have found  
marvelous applications,  
as, for example,  
Boolean \* Algebra (used in Logic),  
by means of which  
you can test, for instance,  
the consistency of  
a whole set of legal statements  
by expressing them in  
“algebraic” form,  
in Boolean Algebra,  
and applying to these expressions  
the rules of manipulation of  
this Algebra!

The implications of this tool  
as an aid to clear thinking  
for general “life situations”  
are not yet fully appreciated! \*\*

And now to give you an idea of  
an Algebra entirely different from  
the one to which you are accustomed,

\* Boole first started this idea (1850).

\*\* See papers on Logic by  
Alonzo Church of Princeton University  
(Galois Institute Press, 1942).

we shall give you here  
a little FINITE ALGEBRA  
constructed by  
Professor Emeritus E. V. Huntington of  
Harvard University.\*

This little Algebra has as its  
basic POSTULATES  
nearly all the postulates of  
ordinary Algebra,  
EXCEPT ONE;  
and yet  
it has only NINE numbers in it:

0, 1, 2, 3, 4, 5, 6, 7, 8.

But,  
you must not think of these as  
being the ordinary numbers  
which you know;  
think of them rather as  
nine SYMBOLS  
which you are to manipulate  
according to certain rules—  
as if you were learning to play  
some new parlor game.

Thus we shall give you  
two tables in which you may  
look up  
the “sum” or “product” of  
any two of the numbers:

\* “The Fundamental Propositions  
of Algebra” (Galois Institute Press, 1941).

SUM TABLE

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

PRODUCT TABLE

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	1	6	8	7	3	5	4
3	0	3	6	4	7	1	8	2	5
4	0	4	8	7	2	3	5	6	1
5	0	5	7	1	3	8	2	4	6
6	0	6	3	8	5	2	4	1	7
7	0	7	5	2	6	4	1	8	3
8	0	8	4	5	1	6	7	3	2

Here we have

$$2 + 2 = 1$$

$$7 + 1 = 8$$

etc.

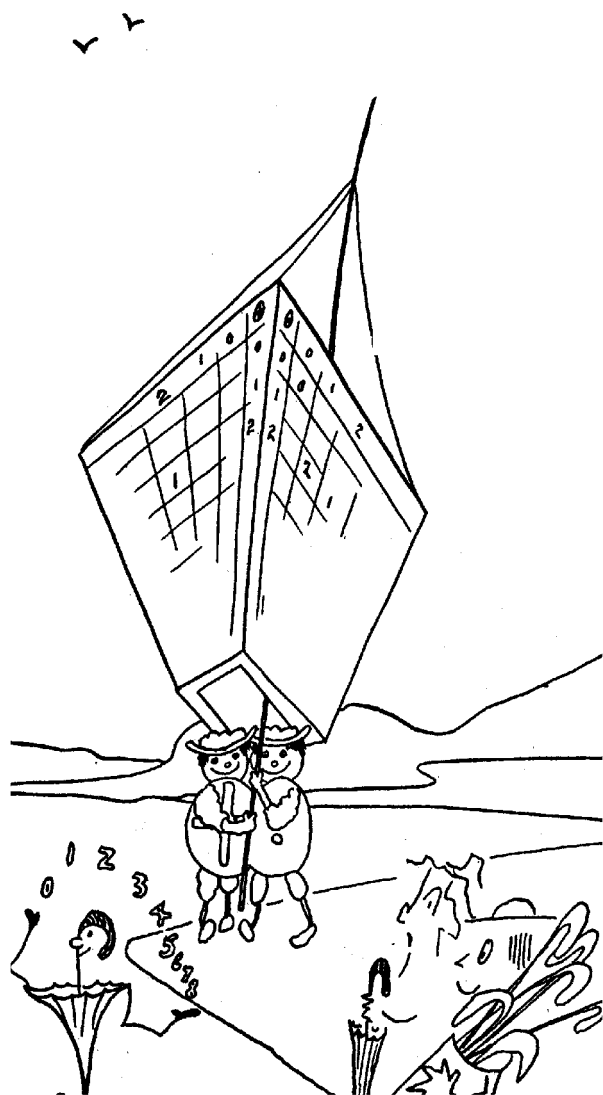
And

$$5 \times 7 = 4$$

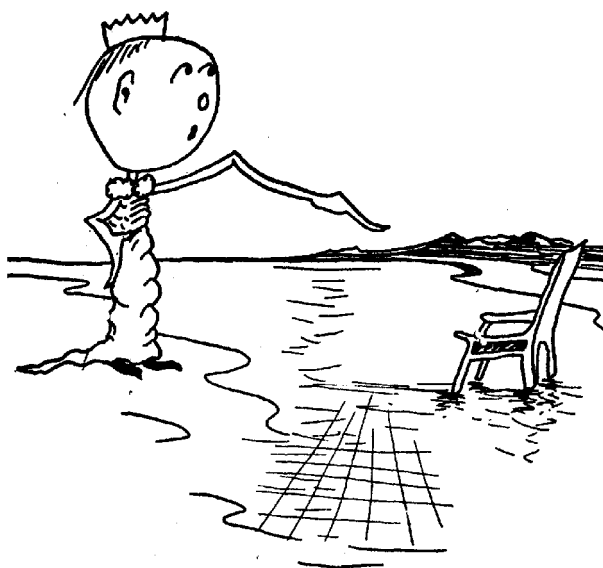
$$2 \times 2 = 1$$

$$8 \times 0 = 0$$

etc.



What interests us here is  
the fundamental idea that  
various Algebras,  
like various Geometries,  
are possible;  
that twice two  
may be four or not four,  
depending upon  
the Algebra in-question;  
that all these  
Algebras and Geometries are  
MAN-MADE;  
that, therefore,  
there is nothing ABSOLUTE  
about any one of them;  
that none of them represents  
THE truth;  
and yet  
all or many of them are  
extremely useful;  
that Man,  
even though he has not found,  
and probably never will find,  
THE truth,  
yet  
he can, by his ability to  
THINK,  
do very well for himself,  
if he would only use it!



This does not mean that  
Man can say to God:  
"See, I am as good as You are.  
I really do not need You at all.  
I can get on very well with  
my own REASONING POWER."

Not at all!

We maintain, on the contrary,  
that  
Man is so far from being  
as good as God  
that he will probably NEVER know  
THE truth:

this emphasizes  
Man's HUMILITY rather than  
his ARROGANCE!

Thus  
since Man has only  
his OWN REASON,  
and NOT God's,  
let him use it  
to the best of his ability,  
and he will get some  
very respectable results,  
but let him never brag that  
he "knows" THE truth!



The Moral: At the end of Chapter VII  
of Part I  
we said:  
"Be a man—not a mouse!"  
And now we add  
"Be a man—but  
do not try to play God!"  
In short, T. C.,  
"BE YOURSELF!"

## XVII. ABSTRACTION—MODERN STYLE

You remember that  
we pointed out on page 82  
the importance of ABSTRACTION,  
and we promised there  
to say more about  
ABSTRACTION as practiced by  
the MODERNS.

Well,  
now that you have seen  
some strange new  
Algebras and Geometries,  
you are prepared to enjoy  
a more abstract system:  
Here, instead of having  
points or numbers as our  
ELEMENTS,  
we shall take the four "objects":  
"chalk," "red," "chair," "desk."  
The sum and product of  
any two of these elements  
may be obtained from

the following two  
tables:

SUM TABLE

	Chalk	Red	Chair	Desk
Chalk	Chalk	Red	Chair	Desk
Red	Red	Chair	Desk	Chalk
Chair	Chair	Desk	Chalk	Red
Desk	Desk	Chalk	Red	Chair

MULTIPLICATION TABLE

	Chalk	Red	Chair	Desk
Chalk	Chalk	Chalk	Chalk	Chalk
Red	Chalk	Red	Chair	Desk
Chair	Chalk	Chair	Chalk	Chair
Desk	Chalk	Desk	Chair	Red

Thus  $\text{Chair} + \text{Red} = \text{Desk}$   
and  $\text{Red} \times \text{Chalk} = \text{Chalk}$   
etc., etc.

Of course the words  
chalk, red, add, multiply, etc.,  
do not have the ordinary meanings,  
but are manipulated in accordance

with whatever rules or postulates  
we have chosen.

In order not to confuse the reader,  
abstract addition and multiplication  
are sometimes designated by

$\oplus$  and  $\otimes$

to distinguish them from  
ORDINARY addition and multiplication,  
for which we use

$+$  and  $\times$

without any circles around them.

Of course  $\oplus$  and  $\otimes$  do not have  
a SPECIFIC meaning,

and therefore give to  
the modern mathematician  
greater FREEDOM to  
INVENT and DEVELOP  
all kinds of systems.

And some of these systems  
may turn out to be of great  
practical value if and when  
they are applied to  
definite problems.

But the mathematician's job is  
to have them ready  
in abstract form so that  
they may be used in any situation  
wherever they may be helpful.

And so you see that  
an important modern trend in  
Mathematics  
is toward more and more  
ABSTRACTION.  
And you can see what a  
source of  
FREEDOM and POWER it is.

In Chapter XX  
you will see how  
this same trend toward  
abstraction  
vitalizes and enriches  
Modern Art as well.

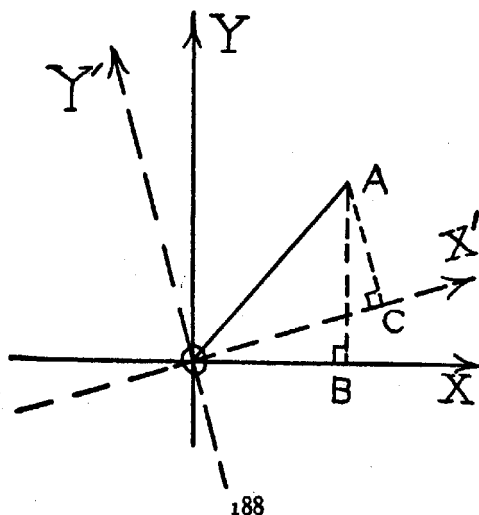
The Moral: Go MODERN:  
learn to  
APPRECIATE THE ABSTRACT.

## XVIII. THE FOURTH DIMENSION

"Very well," you will say,  
"I am quite willing to grant now  
that there may be  
various Geometries and  
various Algebras  
provided we start with  
various assumptions and  
give some new meanings to old words.  
But I still feel that  
there is such a thing as  
THE truth:  
I admit that  
'Twice two is four' is  
NOT such a good sample of it  
as I once thought.  
But how about  
the actual physical world?  
Surely here we are not so free  
to make any assumptions we please.  
Here we must be governed not only  
by clear mathematical thinking—  
which forbids only one thing:  
self-contradiction—

but, in the physical world  
we are also held down by  
**OBJECTIVE FACTS**,  
which I still regard as  
**THE truth!**"

And, since you are quite an  
educated T.C.,  
you may proceed as follows:  
"Suppose, for instance, that  
someone,  
let us call him Mr. K,  
wishes to measure  
the distance from O to A:



and suppose that, for some reason,  
he cannot measure it directly,  
but is obliged to do it indirectly  
(as in so many problems in  
Trigonometry),

by measuring  $OB$  and  $AB$  instead,  
using the axes  $X$  and  $Y$ .

He can now calculate  $OA$  by using  
the Pythagorean Theorem:

$$OA = \sqrt{(OB)^2 + (AB)^2}.$$

Further,

suppose another observer, Mr.  $K'$ ,  
finds it convenient to use instead  
the axes  $X'$  and  $Y'$ .

He then measures  $OC$  and  $AC$   
(instead of  $OB$  and  $AB$  as Mr.  $K$  did)  
and calculates  $OA$  by means of

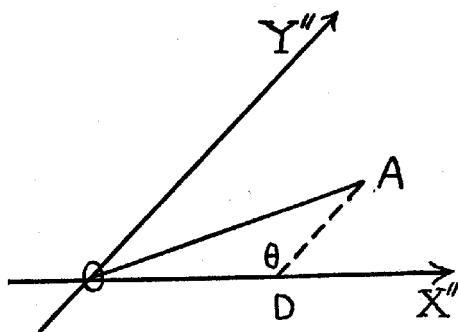
$$OA = \sqrt{(OC)^2 + (AC)^2}.$$

But note well that  
although  $K$  and  $K'$  have made  
DIFFERENT MEASUREMENTS,  
yet they get the  
SAME ANSWER!

And," continues T.C. who  
has really been impressed by  
Part I and sees  
the possibilities,  
"and, mind you,  
if a still more  
individualistic gentleman,  
Mr.  $K''$ ,



prefers to use the axes  $X''$  and  $Y''$   
shown below:



he can still calculate  $OA$ ,  
(although he now measures  $OD$  and  $AD$ )  
by using a well-known formula from  
Trigonometry:

$$AO = \sqrt{(OD)^2 + (AD)^2 - 2 (OD) (AD) \cos \theta},$$

and AGAIN gets the SAME ANSWER!

Thus I conclude that

in spite of the individualism of

$K$  and  $K'$  and  $K''$  and others,

still they can all

'do business' together

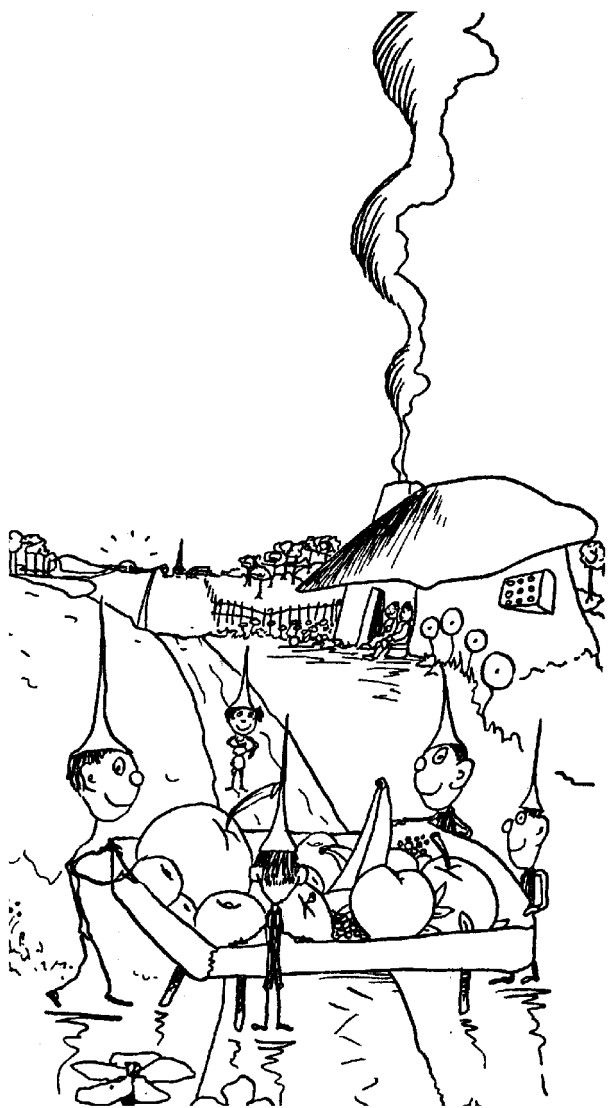
because they agree on the result,

namely, in this case,

the length of  $OA$ ,

which I therefore believe to be

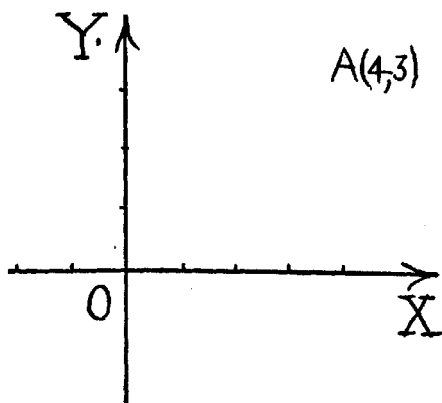
an objective fact.



And it is for this reason that  
I believe in the possibility of  
The Good-Neighbor Policy,  
in which various people can have  
a certain amount of individualism  
and yet can AGREE  
on certain FACTS."

Well, T.C.,  
we agree with nearly everything  
you said,  
but  
we still maintain that  
the human race does not,  
and probably cannot,  
"know the facts."  
And yet  
your idea of  
The Good-Neighbor Policy  
is still acceptable:  
In order to show clearly  
what our position is,  
we must make a little detour  
to discuss what is known as  
"Dimensionality."

As you know so well,  
a point in a plane may be  
designated by a pair of numbers:



Thus point A is designated by  $(4,3)$  because it is reached by going four units to the right of  $O$  and three units up.

And similarly  
for any other point in the plane.

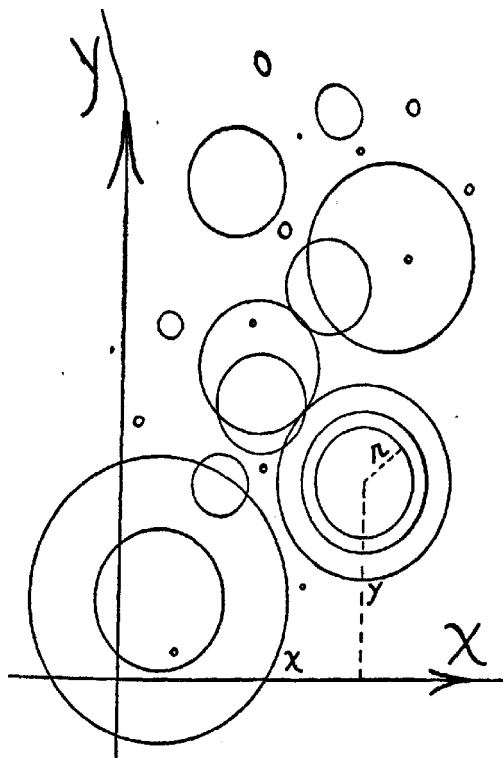
We therefore say that  
a plane is a "two-dimensional space."

Note also that  
it takes 2 numbers to locate  
a point on the surface of a globe,  
namely,  
its latitude and longitude;  
and, therefore,  
the SURFACE of a globe is

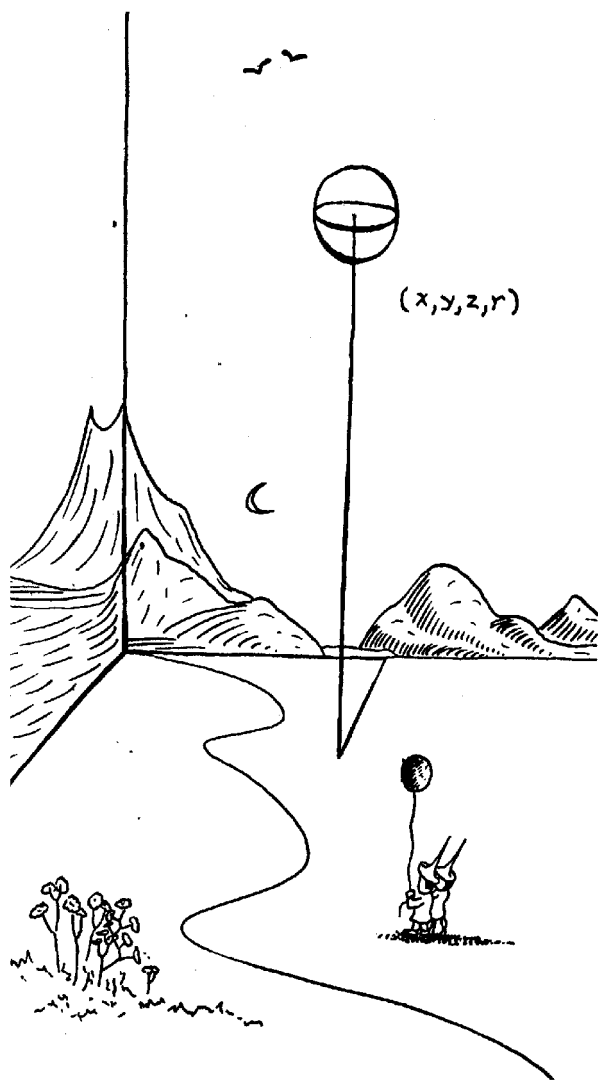
also two-dimensional.  
And similarly for  
*ANY SURFACE*,  
no matter what its shape may be.  
And now, as you also know,  
in three-dimensional space  
it takes 3 numbers to  
locate a point;  
thus, for example, we must give  
the latitude, longitude, and altitude,  
in order to locate a point in  
the actual world we live in.

We must call your attention here  
to an important idea:  
In the above discussion we have been  
talking about locating  
a "point";  
but suppose we choose some other  
"element,"  
instead of "point,"  
say, "circle,"  
and imagine any space which  
we examine  
as filled with circles of  
different sizes,  
with various centers.  
If we now wish to discuss  
the "dimensionality" of  
the space in question,

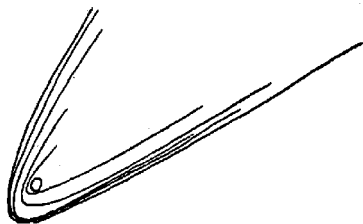
we should have to proceed  
as follows:  
Take first an *ordinary*  
Euclidean plane,



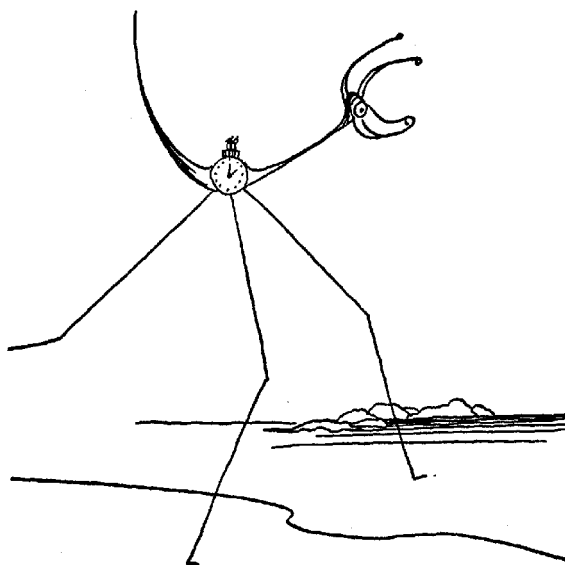
and imagine it to be covered with  
circles instead of points,  
as mentioned above;  
now, to locate  
any particular circle,  
we should first have to direct you  
to its center,  
which would require 2 numbers,  
and then,  
to select this particular circle  
from all the circles of  
different sizes which  
have that same center,  
we should have to give you a  
THIRD number,  
namely, the radius.  
Thus, from this point of view,  
an ordinary Euclidean plane is  
THREE-DIMENSIONAL!  
And, similarly,  
the ordinary "three-dimensional"  
world we live in is  
FOUR-DIMENSIONAL if we use  
spheres instead of points as  
the "elements."  
Thus the "dimensionality" of  
a space  
depends upon the elements chosen.  
But there is of course







c



no confusion here,  
provided we SPECIFY the elements.

Now it has been found convenient  
in modern Physics  
to use "events" instead of "points"  
as the elements in describing  
physical phenomena.  
And since every event is  
characterized by  
FOUR numbers,  
namely, the  
latitude, longitude, altitude, and  
TIME of its occurrence,  
we may therefore speak of  
living in a  
FOUR-DIMENSIONAL WORLD  
without being either  
confused or mystical!

Now let us see what bearing  
this has on the discussion at  
the beginning of this chapter.

## XIX. PREPAREDNESS

As we have already seen  
(see page 189),  
the length of a line on  
a Euclidean plane,  
using any rectangular axes,  
is given by the formula

$$d = \sqrt{x^2 + y^2}.$$

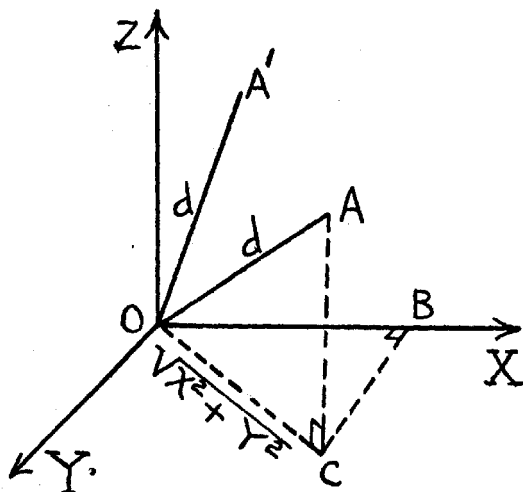
And, similarly,  
in three-dimensional  
Euclidean space

$$d = \sqrt{x^2 + y^2 + z^2}$$

where  $x = OB$ ,  $y = BC$ ,  $z = AC$   
(see the diagram on page 201).

Note that in this  
three-dimensional space,  
 $x^2 + y^2$  taken alone,  
without the  $z$ ,  
is no longer the same for  
different sets of  
rectangular axes.

Thus, if instead of  
changing the axes, we move  $OA$   
to a new position,



such as  $OA'$ ,  
we see that

$$\sqrt{x^2 + y^2} \text{ (or } \sqrt{(x')^2 + (y')^2})$$

is merely the  
"projection" or "shadow"  
of  $OA$  (or  $OA'$ ),

and of course  
the SHADOW of an object  
may CHANGE in length as  
the object is moved around,  
without changing the  
length of the object itself.

Now similarly  
 it has been found in  
 modern Physics  
 that  
 if you take the  
 "interval" between two EVENTS,  
 O and A  
 (instead of two POINTS)  
 in our four-dimensional world,  
 the thing that remains constant is

$$\sqrt{x^2 + y^2 + z^2 + \tau^2}$$

where  $\tau$  is related to  
 the fourth number,  
 the TIME (see page 199).  
 And that

$$\sqrt{x^2 + y^2 + z^2}$$

is no longer a constant,  
 just as

$\sqrt{x^2 + y^2}$  did not remain constant  
 when going from 2 to 3 dimensions  
 (see page 200).

Translated into plain English  
 this says that  
 two observers, K and K',  
 who are moving  
 relatively to an object with  
 different but uniform velocities,  
 do NOT both get the same result for  
 the LENGTH of that object:

that the length of an object is,  
you might say,  
just a three-dimensional  
"projection" or "shadow"  
of a four-dimensional "interval"  
(see pages 201 and 202).  
"Well," replies T.C.,  
"you are beginning to  
sound a little mystical,  
and yet  
when I look at the formulas  
you gave me,  
I follow you all right.  
But, after all,  
to go back to the  
question raised on page 192,  
all you have really done is  
merely to say that  
in modern Physics  
you no longer regard  
the length of an object as  
an immutable fact,  
independent of the particular observer  
(as it was regarded before Einstein),

If you want to know more about  
this interesting point,  
see  
"The Einstein Theory of Relativity,  
the Special Theory"  
by Hugh Gray and Lillian R. Lieber  
(Galois Institute Press).

but, anyway,  
your four-dimensional 'interval'

$$\sqrt{x^2 + y^2 + z^2 + t^2}$$

IS THE SAME

for K and K',

so THIS is now the  
OBJECTIVE FACT  
instead of

$$\sqrt{x^2 + y^2 + z^2}.$$

The principle, however,  
is still the same:  
there ARE physical facts,  
and we are gradually  
finding them out."

We see, T.C.,  
that you are intelligent but  
nineteenth-century minded;  
for,  
the modern twentieth-century  
physicist  
realizes now that  
ANY "facts" that he finds  
are tentative.  
They represent the best we have  
in the light of  
all known observations and  
experiments,  
at a given time.

But he sees clearly that  
even these observations  
are Man's observations,  
subject to the limitations of  
his senses and  
his mind,  
and should in no way be regarded  
as "true."

As Einstein says:  
"Alles was wir machen ist falsch."  
"Everything we make is false."

What then is the use of it?

The obvious answer is:  
"The proof of the pudding is  
in the eating!"

That is,  
if our Science enables us  
to get around more easily in  
this complicated world,  
even if it is no more than  
a mere system of "bookkeeping"  
or a mere "mnemonic,"  
it serves to correlate our  
various observations  
so that we can at least  
remember them and  
envisage them better.



In short,  
the modern physicist,  
making his observations  
in his human way,  
finds it convenient to take as  
postulates for Physics  
certain things which he  
repeatedly observes,  
and then  
he develops by means of  
his human logic  
certain consequences of  
these postulates.  
And, finally,  
he makes more observations in order  
to see whether  
he will OBSERVE these consequences  
which he arrived at by his logic.  
If he does,  
he calls his theory good,  
but he is under no delusion that  
the theory will remain good  
in the light of  
ALL future observations.  
Naturally so long as he does  
observe what he predicted,  
he feels good,  
and we cannot begrudge him  
this feeling of satisfaction.  
For it is only fair to admit that

if we compare  
a scientist's predictions,  
say Einstein's,  
with those of other people,  
we cannot help but be  
**IMPRESSED** with  
the much greater success which  
he has.

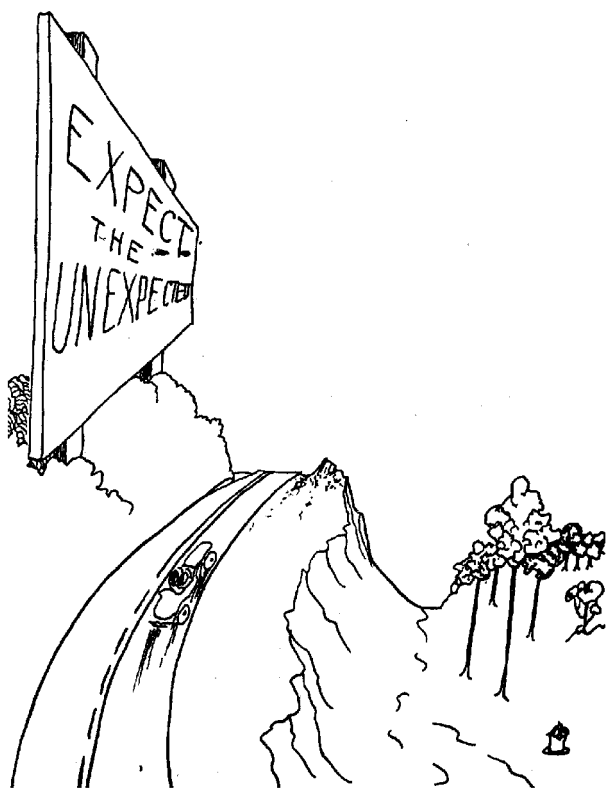
Thus when Einstein predicted in 1916  
that  
if on such and such a date (1919)  
you should go to such and such a place (Africa)  
and set up your camera  
and take a picture of the stars,  
you would find that their positions  
on the photographic plate  
had shifted from their  
normal positions  
by a certain tiny amount  
(about 1.75 seconds of arc).  
And when the scientists followed  
his directions,  
they found just what he predicted!

If anyone outside of Science  
can match this power of prediction  
we shall admit that  
he has as good an approach as  
the scientific one!  
But we need hardly say that

this power of prediction  
has NOT been matched  
by the average  
science-heckling,  
loose-thinking  
loud-speaker!  
And that is why  
we claim that  
scientific predictions are  
a triumph of  
CLEAR THINKING  
even if they are not  
THE ABSOLUTE TRUTH.

And so,  
a modern scientist no longer speaks of  
“objective facts,”  
but of  
“invariants under transformations.”  
Thus

$\sqrt{x^2 + y^2}$   
is an INVARIANT under a  
ROTATION OF AXES in a  
TWO-dimensional Euclidean space,  
but it is  
NOT an invariant under  
such a transformation of axes in  
THREE-dimensional Euclidean space  
(see page 202).  
And you will no doubt agree



that this is a  
more precise  
as well as a more modest  
way of speaking.

And,  
in this way,  
the scientist also holds himself  
in readiness for change!  
For if new observations are made,  
or if he reconsiders his  
basic ideas,  
thus requiring the introduction of  
new transformations,  
he will expect to give up  
his old invariants for new ones.  
And, being prepared for  
this possibility,  
he will not be as startled by  
changes  
as were the 19th century physicists  
when Einstein introduced  
his new system.  
So that now,  
not only has this new system  
been accepted because  
it is more adequate than  
the old one,  
but the physicists have,  
as a result of all this,

a much more wholesome outlook  
on their entire activity.

The Moral: The modern viewpoint  
demands  
greater flexibility of  
mind and  
preparedness for change  
Pull your mind out of  
those muddy old ruts!  
And adapt yourself to  
a continually  
CHANGING world.

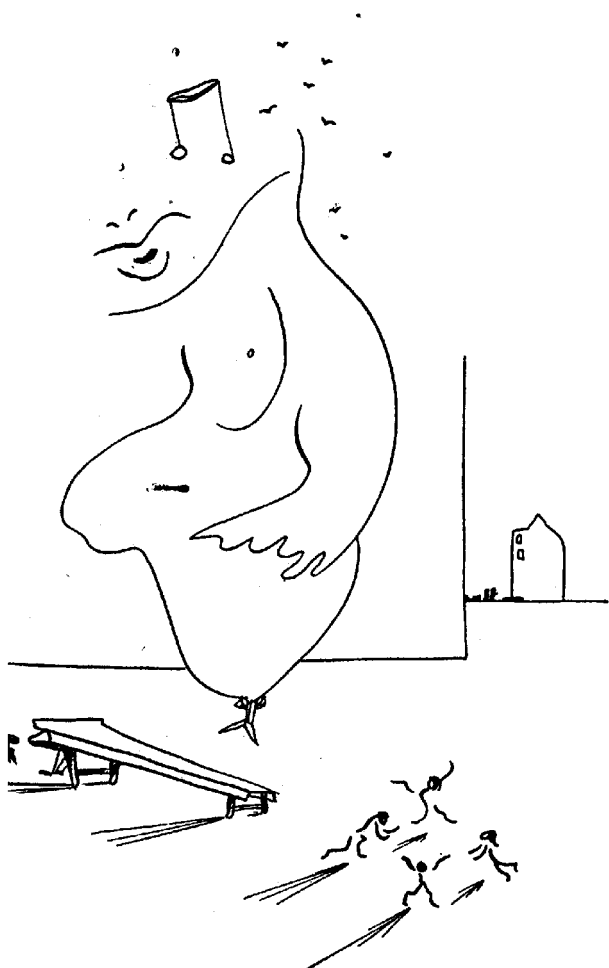
## XX. THESE MODERNS

Let us now return to  
the top floor of the Totem Pole,  
where twice two  
may or may not be four,  
where triangles,  
as well as ladies,  
are dismembered—  
in short,  
where you find  
the MOST MODERN in  
Mathematics, Art, Music, etc.

And let us see what  
these DIFFERENT domains have  
IN COMMON,  
that makes them all MODERN,  
though they seem so unlike each other  
on the surface.

The modern trends seem to be:

(1) Man has begun to realize that





he is a very CREATIVE animal.  
He is growing much bolder  
and venturing out further  
from his old playgrounds.

(2) There is consequently  
INFINITELY MORE VARIETY  
now than heretofore.

(3) In the course of his wanderings,  
he is finding some  
very STRANGE things,  
but he is learning to be  
LESS AFRAID of strangeness.

(4) He is becoming more and more interested  
in ABSTRACT things.

All this you have been realizing  
in connection with Mathematics  
throughout this little book.  
But if you stop to consider  
Modern Music or  
Modern Aviation or  
Modern Art or  
any other Modern domain,  
you will see that there, too,  
these characteristics appear.  
And since you have become  
somewhat familiar with

strange new things,  
and have grown to like them  
(we hope),  
you will doubtless look  
with wonder and admiration  
at the drawings in this chapter.  
For a MODERN EDUCATION must include  
at least a little familiarity  
with the moderns in various domains.

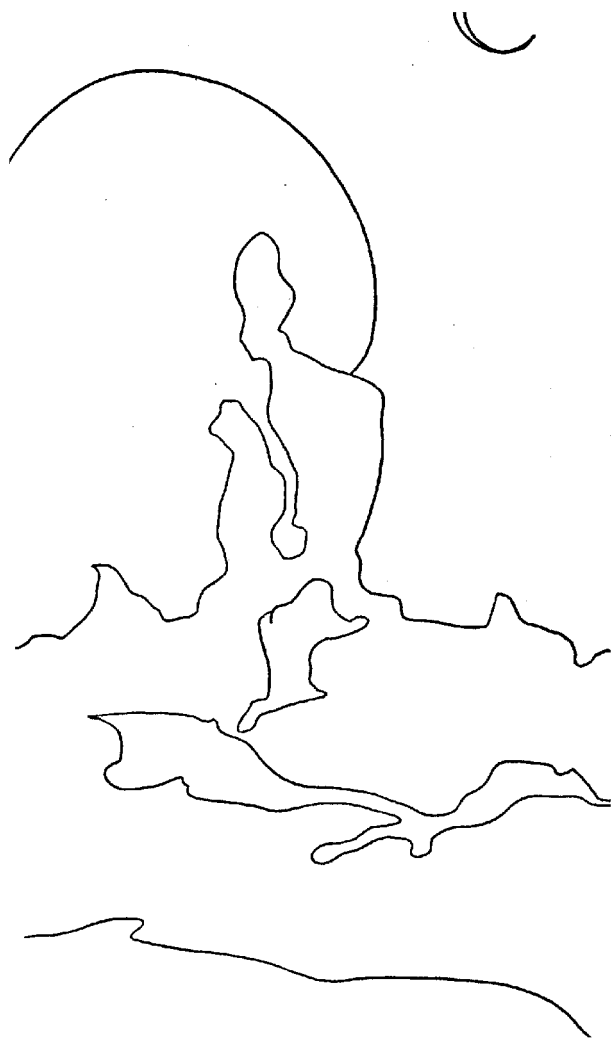
You may expect that  
these drawings  
will be strange and dismembered,  
but you now realize that  
strangeness is a characteristic  
of modernism—  
even in Mathematics and Physics.

And you will not be surprised  
to find that  
modern art has  
INFINITELY MORE VARIETY  
than old-fashioned art,  
just as Mathematics today has  
INFINITELY MORE VARIETY  
than it used to have.

And, above all,  
bear in mind the admonition  
we gave you on page 71,

when we asked you  
not to inquire of  
ANY top-floor man,  
whether mathematician or artist,  
“What is the practical use of  
what you are doing?”  
“What does this mean for  
the Average Man?”  
For, as we told you,  
nobody knows!  
The products of the top floor  
are natural phenomena,  
the most interesting of  
all human documents—  
and if they ever  
come back as a  
first-floor gadget,  
this is NOT  
the most interesting thing  
about them!



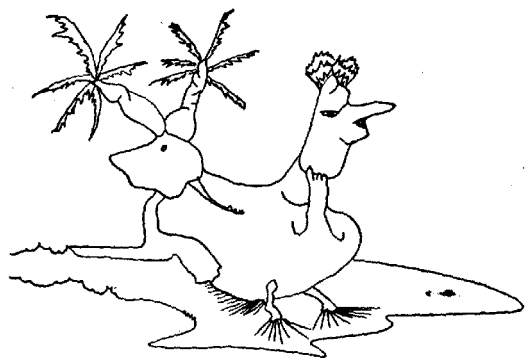
















## THE MORAL

It is possible to have  
agreement  
and yet permit  
different viewpoints  
(Chapters XVIII and XIX).

Unless we compare  
the different viewpoints  
we cannot even speak of  
invariants (page 208) —  
which makes  
isolationism and provincialism  
ridiculous,  
and tolerance essential.

These invariants may be derived  
by various observers  
"With equal rights and equal success" \*  
(page 190).

\* See "Einstein's Theory of Relativity" by H. G. and  
L. R. Lieber (Galois Institute Press).

But what is it that  
each observer has  
the RIGHT to do?  
Obviously ONLY to do his BEST  
(judged by strict standards) :  
To measure as accurately as  
possible  
(as judged by the best  
laboratory practice) ;  
to think straight  
(as judged by the best standards of  
modern mathematicians and logicians) ,  
and NOT MERELY TO HECKLE!

Modesty and humility  
and self-reliance (page 181)  
should characterize man's activity.

Since his knowledge is  
only tentative (page 204)  
he must be  
PREPARED FOR CHANGE (page 210) .

But he must progress with  
a minimum of upheaval (page 162) ,  
respecting tradition without  
being a slave to it (page 117) .

Clear thinking combined with  
careful observation are his  
most "practical" weapon (page 206) .

"Common sense" can be  
enlarged and developed  
and should not remain  
childish (page 93) .

"Human nature" is  
NOT synonymous with  
"money-grabbing" and "throat-cutting":  
Man is a much more  
complex and interesting creature  
(pages 67 and 214) .

War is not to be blamed on Science  
(Chapters V and VI) .

We can have  
freedom without anarchy (page 174) .

Democracy is  
essential to  
human accomplishment  
(page 64 and Chapters XVIII and XIX) .  
But we must be loyal to  
its basic principles or  
we cannot have it at all  
(page 149) .

And so on and so on.

No doubt you can find  
many more morals of this kind  
in Mathematics and Science and Art,  
for this little book is  
only a small sample of  
this point of view  
from which we consider  
not so much the techniques  
(which here are only incidental)  
as the general methods of  
Man's successful accomplishment.  
Perhaps we can learn from them  
how to be equally successful  
in thinking about  
the social sciences, for instance.  
For surely,  
Man, with so much  
ingenuity and originality,  
will not let  
his social problems  
lick him!  
**BUT THEY WILL NOT SOLVE THEMSELVES!**  
He must allow his imagination  
greater freedom,  
as the mathematicians  
and scientists  
and artists  
do;  
and, at the same time,  
must bear in mind

the limitations of his freedom.

And now please turn back to  
the Introduction  
and read it again,  
and consider it  
thoughtfully  
in the light of what  
you have read in  
this little book.

Do you agree with us that  
this material really helps to  
clarify the meanings of  
these concepts?



## SUGGESTED READING

- E. T. BELL: *The Development of Mathematics* (McGraw-Hill).
- G. BOOLE: *The Laws of Thought* (Macmillan).
- EINSTEIN and INFELD: *The Evolution of Physics* (Simon and Schuster).
- MICHAEL FARADAY: *Experimental Researches in Electricity* (Everyman's Library).
- S. I. HAYAKAWA: *Language in Action* (Harcourt, Brace).
- L. T. HOGGEN: *Mathematics for the Million* (Norton).
- E. V. HUNTINGTON: "The Fundamental Propositions of Algebra" (reprinted by the Galois Institute of Mathematics Press from *Monographs on Modern Mathematics* published by Longmans, Green).
- KASNER and NEWMAN: *Mathematics and the Imagination* (Simon and Schuster).
- C. J. KEYSER: *The Human Worth of Rigorous Thinking* (Scripta Mathematica Library).
- : *Mathematical Philosophy, a Study of Fate and Freedom* (Dutton).
- H. G. and L. R. LIEBER: *The Einstein Theory of Relativity*.
- : *Galois and the Theory of Groups*.
- : *Non-Euclidean Geometry*.
- (Galois Institute of Mathematics Press.)
- L. L. THURSTONE: *The Vectors of Mind* (University of Chicago Science Series).
- J. W. YOUNG: *Fundamental Concepts of Algebra and Geometry* (Macmillan).
- Mathematico-Deductive Theory of Rote Learning* (Institute of Human Relations, Yale University Press).
- Papers on selected topics of modern mathematics (Galois Institute of Mathematics Press).









